Lecture 16

<u>Definition</u>: Let X and Y be complex Banach spaces. Let $A : X \to Y$ be a bounded linear operator. Suppose that ① $R(A)$ is closed in Y,

② $\dim N(A) < \infty$,

③ $\dim N(A^t) < \infty$.

Then we call $A : X \to Y$ a <u>Fredholm operator</u>. If $A : X \to Y$ is a Fredholm operator, then the <u>index</u> $i(A)$ of A is defined as

$$i(A) = \dim N(A) - \dim N(A^t).$$

<u>Example</u>: Let X be a complex Banach space. Let $K : X \to X$ be a compact operator. Then $I - K : X \to X$ is Fredholm and $i(I-K) = 0$. This is the <u>Riesz theory</u> of compact operators.

<u>Theorem</u> (F.V. Atkinson) Let X and Y be complex Banach spaces. Let $A : X \to Y$ be a bounded linear operator. Then $A : X \to Y$ is Fredholm if and only if we can find a bounded linear operator $B : Y \to X$ such that

$$BA = I - K_1,$$

$$AB = I - K_2,$$

where $K_1 : X \to X$ and $K_2 : Y \to Y$ are compact operators.

<u>Theorem</u>: Let X and Y be complex Banach spaces. Let $A: X \to Y$ be a bounded linear operator. Suppose A is injective. Then the range $R(A)$ of A is closed in $Y \iff \exists C > 0 \ni$

$$\|x\|_X \leq C \|Ax\|_Y, \quad x \in X.$$

<u>Proof</u>: Suppose that $R(A)$ is closed in Y. Then $R(A)$ is a complex Banach space with norm from that of Y. $\therefore A: X \to R(A)$ is a bijective and bounded linear operator. By the BIV Theorem, $A^{-1}: R(A) \to X$ is a BIO. $\therefore$ there exists a positive constant C such that

$$\|A^{-1}y\|_X \leq C\|y\|_Y, \quad y \in R(A).$$

$\therefore$ for all $x \in X$,

$$\|A^{-1}(Ax)\|_X \leq C\|Ax\|_Y, \quad x \in X.$$

$$\therefore \|x\|_X \leq C\|Ax\|_Y, \quad x \in X.$$

Conversely, suppose that there exists a positive constant C such that

$$\|x\|_X \leq C\|Ax\|_Y, \quad x \in X.$$

Let $\{y_k\}_{k=1}^\infty$ be a sequence in $R(A)$ such that $y_k \to y$ in Y. Let $x_k \in X$ be such that $Ax_k = y_k$. Then

$$\|x_j - x_k\|_X \leq C\|Ax_j - Ax_k\|_Y = C\|y_j - y_k\|_Y \to 0$$

as $j, k \to \infty$. $\therefore \{x_j\}_{j=1}^\infty$ is a Cauchy sequence $\therefore x_j \to x$ for some x in X as $j \to \infty$. $Ax_j \to Ax$ in Y as $j \to \infty$. $\therefore y_k \to Ax$ in Y as $k \to \infty$. $\therefore y = Ax$ $\therefore y \in R(A)$. $\therefore R(A)$ is closed in Y.

We now work in Hilbert spaces

Theorem: Let X, Y, Z be complex, separable and infinite-dimensional Hilbert spaces. Let $A_1 : X \to Y$ and $A_2 : Y \to Z$ be Fredholm operators. Then $A_2 A_1 : X \to Z$ is Fredholm and
$$i(A_2 A_1) = i(A_2) + i(A_1).$$

Proof: ($A_2 A_1 : X \to Z$ is a Fredholm operator)

$\left.\begin{array}{l} A_1 : X \to Y \\ A_2 : Y \to Z \end{array}\right\}$ are Fredholm.

So $B_1 A_1 = I - K_1$, $B_1 \in B(Y, X)$, $K_1 \in K(X)$

$A_1 B_1 = I - K_2$, $K_2 \in K(Y)$,

$B_2 A_2 = I - K_3$, $B_2 \in B(Z, Y)$, $K_3 \in K(Y)$,

$A_2 B_2 = I - K_4$, $K_4 \in K(Z)$.

$$\therefore (B_1 B_2)(A_2 A_1) = B_1(I - K_3)A_1 = B_1 A_1 - B_1 K_3 A_1$$
$$= I - K_1 - B_1 K_3 A_1$$
$$= I + K_4, \quad K_4 \in K(X)$$

$$(A_2 A_1)(B_1 B_2) = A_2(I - K_2)B_2 = A_2 B_2 - A_2 K_2 B_2$$
$$= I - K_4 - A_2 K_2 B_2 \in K(Z)$$

$\therefore$ by Atkinson's theorem, $A_2 A_1$ is Fredholm.