Lecture 15

<u>Theorem</u>: Let $\sigma \in S^{2m}$, $m > 0$, be an elliptic symbol such that we can find a positive constant C and a constant λ_0 for which

$$\operatorname{Re}(-T_\sigma \varphi, \varphi) \geq C\|\varphi\|_{m,2}^2 - \lambda_0 \|\varphi\|_2^2, \quad \varphi \in \mathcal{S}.$$

Then $T_{\sigma,0}$ is the infinitesimal generator of a one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$.

<u>Lemma</u>: Let $\lambda > \lambda_0$. Then for every $f \in L^2(\mathbb{R}^n)$, there exists a unique solution u in $H^{2m,2}$ of the equation $(\lambda I - T_{\sigma,0})u = f$ on $\mathbb{R}^n$. Moreover,

$$\|(\lambda I - T_{\sigma,0})u\|_2 \geq (\lambda - \lambda_0)\|u\|_2, \quad u \in L^2(\mathbb{R}^n).$$

<u>Proof of Theorem</u>: $T_{\sigma,0} : \mathcal{D}(T_{\sigma,0}) \to L^2(\mathbb{R}^n)$

$$\mathcal{D}(T_{\sigma,0}) \underset{\text{dense subspace}}{\subset} L^2(\mathbb{R}^n)$$

is a closed linear operator. By Lemma, $\lambda I - T_{\sigma,0}$ is bijective. $\therefore R(\lambda; T_{\sigma,0})$ exists for all $\lambda > \lambda_0$. $\therefore \{\lambda \in \mathbb{R} : \lambda > \lambda_0\} \subset \rho(T_{\sigma,0})$. For $\lambda > \lambda_0$,

$$\|R(\lambda; T_{\sigma,0})u\|_2 \leq \frac{1}{\lambda - \lambda_0}\|u\|_2, \quad u \in L^2(\mathbb{R}^n).$$

$$\therefore \|R(\lambda; T_{\sigma,0})\| \leq \frac{1}{\lambda - \lambda_0}, \quad \lambda > \lambda_0.$$

$$\therefore \|R(\lambda; T_{\sigma,0})^n\| \leq \|R(\lambda; T_{\sigma,0})\|^n \leq \frac{1}{(\lambda - \lambda_0)^n},$$

for all $\lambda > \lambda_0$. $\therefore T_{\sigma,0}$ is the infinitesimal generator of a one-parameter semigroups of bounded linear operators on $L^2(\mathbb{R}^n)$.

Theorem: Let $\sigma \in S^{2m}$, $m > 0$, be a strongly elliptic symbol. Then $-T_{\sigma,0}$ is the infinitesimal generator of a one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$.

Proof: By Gårding's inequality, we can find a positive constant C and a constant C_s for all $s \geq \frac{1}{2}$ such that

$$\operatorname{Re}(T_\sigma\varphi, \varphi) \geq C\|\varphi\|_{m,2}^2 - C_s\|\varphi\|_{m-s,2}^2, \quad \varphi \in \mathcal{S}.$$

In fact, C_s can be chosen to be > 0. Let $\varepsilon \in \left(0, \frac{C}{C_s}\right)$

By Exercise 12.6 (Ehrling's Inequality), there exists a positive constant C_ε such that

$$\|\varphi\|_{m-s,2}^2 \leq \varepsilon\|\varphi\|_{m,2}^2 + C_\varepsilon\|\varphi\|_2^2, \quad \varphi \in \mathcal{S}.$$

$$\therefore \operatorname{Re}(T_\sigma\varphi, \varphi) \geq C\|\varphi\|_{m,2}^2 - C_s\varepsilon\|\varphi\|_{m,2}^2 - C_sC_\varepsilon\|\varphi\|_2^2$$
$$= (C - C_s\varepsilon)\|\varphi\|_{m,2}^2 - C_sC_\varepsilon\|\varphi\|_2^2,$$
$$\varphi \in \mathcal{S}.$$

So, $-T_{\sigma,0}$ is the infinitesimal generator of a one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$.

Chapter 20

Let X and Y be complex Banach spaces. Let $A : X \to Y$ be a bounded linear operator. Consider the equation $Ax = y$, where $y \in Y$ is given.

Question: Is it true that for all $y \in Y$, there exists a unique solution x in X?

Answer: No!

Next Best: We want to be able to solve the equation for as many y as possible and the number of solutions for each $y \in Y$ to be as small as possible.

Definition: Let $A : X \to Y$ be a bounded linear operator. Suppose that

- the range $R(A)$ of A is closed in Y,
- $\dim N(A) < \infty$, where $N(A)$ is the null space of A,
- $\dim N(A^t) < \infty$, where $A^t : Y' \to X'$ is the adjoint of A.

Then we call A a Fredholm operator.

Definition: Let $A : X \to Y$ be a Fredholm operator. Then we define the index $i(A)$ of A by

$$i(A) = \dim N(A) - \dim N(A^t).$$