Lecture 13

(Necessity) Suppose that $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a one-parameter semigroup $\{T(t) : t \geq 0\}$ on X. Let $g : [0,\infty) \to \mathbb{R}$ be defined by

$\mathcal{D}(A) \subset X$ dense subspace

$$g(t) = \ln \|T(t)\|, \quad t \geq 0.$$

Then
$$g(s+t) \leq g(s) + g(t), \quad s, t \in [0,\infty).$$

Indeed,
$$\begin{aligned} g(s+t) &= \ln \|T(s+t)\| = \ln \|T(s)T(t)\| \\ &\leq \ln(\|T(s)\|\|T(t)\|) = \ln\|T(s)\| + \ln\|T(t)\| \\ &= g(s) + g(t). \end{aligned}$$

Let $t_0 > 0$. Then for all $t \in [0,\infty)$, we can write $t = nt_0 + s$, where n is an integer depending on t and $0 \leq s < t_0$. So, as $t \to \infty$,
$$\frac{g(t)}{t} \leq \frac{n g(t_0)}{t} + \frac{g(s)}{t} = \frac{g(t_0)}{\frac{nt_0 + s}{n}} + \frac{g(s)}{t} \to \frac{g(t_0)}{t_0}.$$

$$\therefore \overline{\lim_{t\to\infty}} \frac{g(t)}{t} \leq \frac{g(t_0)}{t_0}.$$

So,
$$\overline{\lim_{t\to\infty}} \frac{g(t)}{t} \leq \inf_{t_0 > 0} \frac{g(t_0)}{t_0}.$$

Let $\delta > \inf_{t_0>0} \frac{g(t_0)}{t_0}$. Then
$$\inf_{R>0} \sup_{t>R} \frac{g(t)}{t} < \delta.$$

$\therefore$ there exists a positive number R such that
$$\sup_{t>R} \frac{g(t)}{t} < \delta.$$

$\therefore \dfrac{\gamma(t)}{t} < \delta, \quad t > R.$

$\therefore \dfrac{\ln \|T(t)\|}{t} < \delta, \quad t > R,$

which is

$$\|T(t)\| < e^{\delta t}, \quad t > R.$$

Since $\dfrac{\|T(t)\|}{e^{\delta t}}$ is continuous on $[0, R]$, we see that there is a positive constant C such that

$$\frac{\|T(t)\|}{e^{\delta t}} \leq C, \quad t \in [0, R].$$

$\therefore$ there exists a positive constant $M = \max(1, C)$ such that

$$\|T(t)\| \leq M e^{\delta t}, \quad t \in [0, \infty).$$

Now, let $\lambda > \delta$. Then for all $x \in X$, we define $R(\lambda)x$ by

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t) x \, dt.$$

Then $R(\lambda) : X \to X$ is a bounded linear operator. Indeed, for all $x \in X$,

$$\|R(\lambda)x\|_X \leq M \int_0^\infty e^{-\lambda t} e^{\delta t} \, dt \, \|x\|_X$$

$$= M \int_0^\infty e^{-(\lambda - \delta)t} \, dt \, \|x\|_X .$$

For $h > 0$,

$$A_h R(\lambda)x = \frac{T(h)R(\lambda)x - R(\lambda)x}{h}$$

$\therefore A_h R(\lambda) x$

$$= \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t+h)x\,dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt$$

$$= \frac{1}{h}\int_h^\infty e^{-\lambda(s-h)} T(s)x\,ds - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt$$

$$= \frac{e^{\lambda h}}{h}\int_0^\infty e^{-\lambda s} T(s)x\,ds - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda s} T(s)x\,ds - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt$$

$$= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} T(t)x\,dt$$

$\to \lambda R(\lambda)x - x$ as $h \to 0+$.

$\therefore R(\lambda)x \in \mathcal{D}(A)$ and

$A(R(\lambda)x) = \lambda R(\lambda)x - x$

$\therefore (\lambda I - A)R(\lambda)x = x$.

Now, for all $x \in \mathcal{D}(A)$,

$$A R(\lambda)x = \lim_{h\to 0+}\left(\frac{T(h)-I}{h} R(\lambda)x\right)$$

$$= \lim_{h\to 0+}\left(\frac{1}{h}T(h)\int_0^\infty e^{-\lambda t} T(t)x\,dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt\right)$$

$$= \lim_{h\to 0+}\frac{1}{h}\left(\int_0^\infty e^{-\lambda t}\, T(t+h)x - T(t)x\,dt\right.$$

$$= \lim_{h\to 0+}\int_0^\infty e^{-\lambda t}\,\frac{T(t+h)x - T(t)x}{h}\,dt$$

$= R(\lambda)Ax.$

$\therefore R(\lambda)(\lambda I - Ax) = \lambda R(\lambda)x - R(\lambda)Ax = (\lambda I - A)R(\lambda)x = x.$

$\therefore \lambda \in \rho(A)$, $\lambda > \delta$.