Lecture 12

Definition: For all $x \in X$ and all $t \geq 0$, we define $T(t)x$ by
$$T(t)x = \lim_{\lambda \to \infty} S_\lambda(t)x = \lim_{\lambda \to \infty} e^{tB_\lambda}x.$$

Remark: For all $t \geq 0$, $T(t) : X \to X$ is a bounded linear operator. Indeed, for $t \geq 0$, there exists a real number $\lambda(t)$ such that $\lambda \geq \lambda(t) \Rightarrow$ for all $x \in X$,
$$\|T(t)x\|_X = \left\| \lim_{\lambda\to\infty} e^{tB_\lambda}x \right\|_X = \lim_{\lambda \to\infty} \|e^{tB_\lambda}x\| \leq Me^{t\omega_1}\|x\|$$
where $\omega_1 > \omega$. For all t in a compact subset of $[0,\infty)$, $\underbrace{}_{K=[a,b]}$
there exists a real number $\lambda(b)$ such that
$$\lambda, \mu \geq \lambda(b) \Rightarrow \|S_\lambda(t)x - S_\mu(t)x\|_X \leq Me^{t_0\omega_1}\|B_\lambda x - B_\mu x\|_X$$
for all $t \in [a,b]$. $\therefore$ $S_\lambda(t)x \to T(t)x$ uniformly with respect to t on compact subsets of $[0,\infty)$ as $\lambda \to \infty$ for all x in X. $\therefore$ for all $x \in X$,
$$[0,\infty) \ni t \mapsto T(t)x \in X$$
is continuous. To check the semigroup property, note that for all $x \in X$,
$$T(0)x = \lim_{\lambda\to\infty} S_\lambda(0)x = x.$$
$\therefore$ $T(0) = I$. Now, for all $0 \leq s, t < \infty$, $x \in X$,
$$T(s+t)x = \lim_{\lambda\to\infty} S_\lambda(s+t)x = \lim_{\lambda\to\infty} S_\lambda(s)S_\lambda(t)x.$$

Next,
$$\begin{aligned}
&\|S_\lambda(s)S_\lambda(t)x - T(s)T(t)x\|_X \\
&= \|S_\lambda(s)S_\lambda(t)x - S_\lambda(s)T(t)x + S_\lambda(s)T(t)x - T(s)T(t)x\|_X \\
&\leq \|S_\lambda(s)\|\,\|S_\lambda(t)x - T(t)x\|_X + \|(S_\lambda(s) - T(t))T(t)x\|_X
\end{aligned}$$
$\to 0$ as $\lambda \to \infty$.

Now, A is the infinitesimal generator B of the one-parameter semigroup $\{T(t) : t \in [0,\infty)\}$. Let $x \in X$. Then

$$S_\lambda(t)x - x = \int_0^t \frac{d}{ds} S_\lambda(s)x\,ds = \int_0^t S_\lambda(s) B_\lambda x\,ds.$$

For all $x \in \mathcal{D}(A)$ and all $s \in [0,\infty)$,

$$\|S_\lambda(s)B_\lambda x - T(s)Ax\|_X$$

$$= \|S_\lambda(s)B_\lambda x - S_\lambda(s)Ax + S_\lambda(s)Ax - T(s)Ax\|_X$$

$$= \|S_\lambda(s)\|\,\|B_\lambda x - Ax\|_X + \|S_\lambda(s) - T(s)\|\,\|Ax\|_X$$

$\to 0$ uniformly with respect to s on every compact subset of $[0,\infty)$. $\therefore$

$$T(t)x - x = \int_0^t T(s)Ax\,ds, \quad x \in \mathcal{D}(A),\ t \geq 0.$$

$\therefore$ $$Bx = \lim_{t\to 0+} \frac{T(t)x - x}{t} = \lim_{t\to 0}\int_0^t T(s)Ax\,ds = Ax, \quad x \in \mathcal{D}(A).$$

$\therefore$ B is an extension of A. So we only need to prove that $B = A$. To prove that $\mathcal{D}(A) = \mathcal{D}(B)$, we first claim that there exists a real number $\lambda_1 \in \mathbb{R}$ such that $\lambda \in \rho(B)$ for all $\lambda > \lambda_1$. Then there exists a $\lambda \in \mathbb{R}$ such that $\lambda \in \rho(A) \cap \rho(B)$. $\therefore$

$$(\lambda I - B)\mathcal{D}(A) = (\lambda I - A)\mathcal{D}(A) = X = (\lambda I - B)\mathcal{D}(B)$$

So for all $x \in \mathcal{D}(B)$, there exists an element $z \in \mathcal{D}(A)$ such that

$$(\lambda I - B)z = (\lambda I - A)x = (\lambda I - B)x.$$

$\therefore$ $x = z$.

It remains to prove the Claim. For $\lambda > \omega_1$, we define for all x in X, $R(\lambda)x$, by

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt.$$

It is easy to see that $R(\lambda): X \to X$ is a bounded linear operator. So,

$$B_h R(\lambda)x = \frac{T(h)R(\lambda)x - R(\lambda)x}{h}$$

$$= \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t+h)x\,dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt$$

$$= \frac{1}{h}\int_h^\infty e^{-\lambda(s-h)} T(s)x\,ds - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt$$

$$= \frac{e^{\lambda h}}{h}\int_0^\infty e^{-\lambda s} T(s)x\,ds - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} T(t)x\,dt$$

$$= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} T(t)x\,dx$$

$$\longrightarrow \lambda R(\lambda)x - x \text{ as } h \to 0+.$$

So, for all $\lambda > \omega_1$, $x \in X$,

$$\begin{cases} R(\lambda)x \in \mathcal{D}(B), \\ BR(\lambda)x = \lambda R(\lambda)x - x. \end{cases}$$

$\therefore (\lambda I - B)R(\lambda)x = x.$

Now for $\lambda > \omega_1$, $x \in \mathcal{D}(B)$,

$$BR(\lambda)x = \left(\lim_{h\to 0} \frac{T(h) - I}{h}\right) R(\lambda)x$$

$$= \lim_{h\to 0}\left(\frac{1}{h}T(h)\int_0^\infty e^{-\lambda t} T(t)x\,dx - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\int_0^\infty e^{-\lambda t} T(t+h)x\,dx - \int_0^\infty e^{-\lambda t} T(t)x\,dt\right)$$

$$= R(\lambda)Bx.$$