Lecture 11

Theorem: (Hille-Yosida-Phillips)

Let $A : \mathcal{D}(A) \to X$ be a closed linear operator. Then

$\mathcal{D}(A) \subset X$ dense subspace

A is the infinitesimal generator of a one-parameter semigroup $\{T(t) : t \geq 0\}$ on X $\Leftrightarrow$ we can find a positive number M and a real constant ω such that

$$\{\lambda \in \mathbb{R} ; \lambda > \omega\} \subset \rho(A)$$

and

$$\|R(\lambda; A)^n\| \leq \frac{M}{(\lambda-\omega)^n}, \quad \lambda > \omega.$$

Proof: For $\lambda > \omega$, let

$$B_\lambda = -\lambda(I - \lambda R(\lambda; A)).$$

Then

$$\|e^{tB_\lambda}\| \leq M e^{\frac{t\omega\lambda}{\lambda-\omega}} \to M e^{t\omega} \text{ as } \lambda \to \infty.$$

Fix $\omega_1 > \omega$. Then for every $t \in [0,\infty)$, there exists a positive number $\lambda(t)$ such that

$$\lambda \geq \lambda(t) \Rightarrow \|e^{tB_\lambda}\| \leq M e^{t\omega_1}.$$

Claim: $\lim\limits_{\lambda\to\infty} B_\lambda x = Ax$, $x \in \mathcal{D}(A)$.

Indeed, for all $x \in \mathcal{D}(A)$,

$$\lambda R(\lambda; A)x - x = R(\lambda; A)Ax.$$

$$\therefore \|\lambda R(\lambda; A)x - x\|_X \leq \frac{M}{\lambda-\omega}\|Ax\|_X \to 0$$

as $\lambda \to \infty$. Now,

$$\|\lambda R(\lambda; A)\| \leq \frac{\lambda M}{\lambda-\omega} \to M \text{ as } \lambda \to \infty.$$

$\therefore$ there exists a positive number λ_0 such that

$$\|\lambda R(\lambda; A)\| \leq 2M.$$

Let $x \in X$. Then for every positive number ε, there exists an element z in $\mathcal{D}(A)$ with

$$\|x - z\|_X < \min\left(\frac{\varepsilon}{6M}, \frac{\varepsilon}{3}\right).$$

$\therefore$

$$\|\lambda R(\lambda;A)x - x\|_X$$

$$\leq \|\lambda R(\lambda;A)x - \lambda R(\lambda;A)z\|_X + \|\lambda R(\lambda;A)z - z\|_X + \|z - x\|_X$$

$$< \frac{\varepsilon}{6M}\cdot 2M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for λ sufficiently large. $\therefore$ for all x in X,

$$\lambda R(\lambda;A)x \to x$$

in X as $\lambda \to \infty$. Now, for all $x \in \mathcal{D}(A)$,

$$B_\lambda x = -\lambda(I - \lambda R(\lambda;A))x$$

$$= -\lambda x + \lambda R(\lambda;A)\lambda x - \lambda R(\lambda;A)Ax + \lambda R(\lambda;A)Ax$$

$$= -\lambda x + \lambda x + \lambda R(\lambda;A)Ax \to Ax \text{ in } X \text{ as } \lambda \to \infty.$$

For $\lambda > \omega$, let

$$S_\lambda(t) = e^{tB_\lambda}, \quad t \in [0,\infty).$$

By Exercise 19.4,

$$R(\lambda;A)R(\mu;A) = R(\mu;A)R(\lambda;A).$$

$$\therefore B_\lambda B_\mu = \lambda\mu(I - \lambda R(\lambda;A))(I - \mu R(\mu;A))$$

$$= \lambda\mu(I - \lambda R(\lambda;A) - \mu R(\mu;A) + \lambda\mu R(\mu;A)R(\lambda;A))$$

$$= B_\mu B_\lambda.$$

$\therefore$ for all $\lambda, \mu \in (\omega,\infty)$, $t \in [0,\infty)$,

$$B_\mu S_\lambda(t) = S_\lambda(t) B_\mu.$$

and so,

$$S_\lambda(t)x - S_\mu(t)x$$
$$= \int_0^t \frac{d}{ds}\left(S_\mu(t-s)S_\lambda(s)x\right)ds = \int_0^t S_\mu(t-s)S_\lambda(s)(B_\lambda - B_\mu)x\,ds.$$

So, for each fixed $t \in [0,\infty)$, there exists a real number $\lambda(t)$ such that

$$\lambda, \mu \in (\lambda(t), \infty) \Rightarrow$$
$$\|S_\lambda(t)x - S_\mu(t)x\|_X \le M^2 t e^{t\omega_1}\|B_\lambda x - B_\mu x\|_X.$$

$\therefore$ for every $t \in [0,\infty)$ and every $x \in \mathcal{D}(A)$,

$$\|S_\lambda(t)x - S_\mu(x)\|_X \to 0$$

as $\lambda, \mu \to \infty$. Now for all $t \in [0,\infty)$ and all $x \in X$, for every $\varepsilon > 0$ we can find an element z in $\mathcal{D}(A)$ with

$$\|x - z\|_X < \frac{\varepsilon}{3Me^{t\omega_1}}.$$

So, for all λ, μ in $(\lambda(t), \infty)$,

$$\|S_\lambda(t)x - S_\mu(t)x\|_X$$
$$= \|S_\lambda(t)x - S_\lambda(t)z + S_\lambda(t)z - S_\mu x\|_X$$
$$\le Me^{t\omega_1}\frac{\varepsilon}{3Me^{t\omega_1}} + 2Me^{t\omega_1}\frac{\varepsilon}{3Me^{t\omega_1}} = \varepsilon.$$

$\therefore \lim_{\lambda\to\infty} S_\lambda(t)$ exists for every $t \in [0,\infty)$.

We define $T(t)$ for every $t \in [0,\infty)$ by

$$T(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x, \quad x \in X.$$