Lecture 10

Let X be a complex Banach space. Let $A : \mathcal{D}(A) \longrightarrow X$ be a closed linear operator.

$$\mathcal{D}(A) \underset{\text{dense subspace}}{\subset} X$$

Question: When is A the infinitesimal generator of a one-parameter semigroup $\{T(t) : t \geq 0\}$ on X?

We need some spectral analysis:

Let $\rho(A)$ be the set defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A : \mathcal{D}(A) \to X \text{ is bijective}\}.$$

We call $\rho(A)$ the resolvent set of A. If $\lambda \in \rho(A)$, then the bounded linear operator $(\lambda I - A)^{-1} : X \to X$ is known as the resolvent of A at λ and denoted by $R(\lambda; A)$.

Theorem (Hille-Yosida-Phillips Theorem) Let $A : \mathcal{D}(A) \longrightarrow X$ be a closed linear operator. Then

$$\mathcal{D}(A) \underset{\text{dense subspace}}{\subset} X$$

A is the infinitesimal generator of a one-parameter semigroup $\{T(t) : t \geq 0\}$ on X $\iff$ we can find a positive number M and a real number ω such that

$$\{\lambda \in \mathbb{R} : \lambda > \omega\} \subseteq \rho(A)$$

and

$$\| R(\lambda, A)^n \| \leq \frac{M}{(\lambda - \omega)^n}$$

for all $\lambda > \omega$ and $n \in \mathbb{N}$.

Proof (Sufficiency) For $\lambda > \omega$, we define $B_\lambda : X \to X$ by

$$B_\lambda = -\lambda(I - \lambda R(\lambda; A)).$$

Then for all $t \in \mathbb{R}$,

$$e^{tB_\lambda} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!} R(\lambda; A)^n.$$

$\therefore$

$$\|e^{tB_\lambda}\| = e^{-\lambda t} M \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!(\lambda-\omega)^n}$$

$$= e^{-\lambda t} M e^{\frac{\lambda^2 t}{\lambda-\omega}}$$

$$= M e^{-\lambda t\left(1-\frac{\lambda}{\lambda-\omega}\right)} = M e^{-\lambda t \frac{-\omega}{\lambda-\omega}}$$

$$= M e^{\frac{t\lambda\omega}{\lambda-\omega}} \longrightarrow M e^{t\omega} \text{ as } \lambda \to \infty.$$

So, for all $\omega_1 > \omega$,

$$\|e^{tB_\lambda}\| < M e^{t\omega_1}$$

whenever λ is large enough.

Now, $\lim_{\lambda\to\infty} B_\lambda x = Ax$, $x \in \mathcal{D}(A)$. Indeed, for all x in $\mathcal{D}(A)$,

$$\lambda R(\lambda; A)x - x$$
$$= \lambda R(\lambda; A)x - R(\lambda; A)Ax + R(\lambda; A)Ax - x$$
$$= (\lambda I - A)R(\lambda; A)x + R(\lambda; A)Ax - x$$
$$= x + R(\lambda; A)Ax - x = R(\lambda; A)Ax$$

$$\therefore \|\lambda R(\lambda; A)x - x\| = \|R(\lambda; A)Ax\|$$
$$\leq \frac{M}{\lambda-\omega}\|Ax\| \to 0 \text{ as } \lambda \to \infty.$$

But

$$\| \lambda R(\lambda; A) \| \leq \frac{M\lambda}{\lambda - \omega} \to M \text{ as } \lambda \to \infty.$$

$\therefore \| \lambda R(\lambda; A) \| \leq 2M$ for sufficiently large λ.

Let $x \in X$. Then for all $\varepsilon > 0$, there exists an element z in $\mathcal{D}(A)$ with

$$\| x - z \| < \min\left(\frac{\varepsilon}{6M}, \frac{\varepsilon}{3}\right).$$

$\therefore \| \lambda R(\lambda; A)x - x \|$

$= \| \lambda R(\lambda; A)x - \lambda R(\lambda; A)z + \lambda R(\lambda; A)z - z + z - x \|$

$\leq 2M \| x - z \| + \| \lambda R(\lambda; A)z - z \| + \| z - x \|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ for sufficiently large λ.

So, for all $x \in \mathcal{D}(A)$,

$$B_\lambda x = -\lambda x + \lambda^2 R(\lambda; A)x$$

$= -\lambda x + \lambda R(\lambda; A)\lambda x - \lambda R(\lambda; A)Ax + \lambda R(\lambda; A)Ax$

$\quad\; \lambda R(\lambda; A)Ax - \lambda R(\lambda; A)Ax + Ax$

$\to Ax$

$= Ax$

as $\lambda \to \infty$.