Lecture 1

Recall: Let $m \in \mathbb{R}$. Then S^m is the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}, \quad x,\xi \in \mathbb{R}^n.$$

We call σ a symbol of order m.

Let $\sigma \in S^m$. Then for all $\varphi \in \mathcal{S}$, we define a new function $T_\sigma\varphi$ on $\mathcal{S}$ by

$$(T_\sigma\varphi)(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)\,d\xi, \quad x \in \mathbb{R}^n.$$

We call T_σ the pseudo-differential operator with symbol σ.

Definition: Let $\sigma \in S^m$. Suppose that there exist positive constants C and R such that

$$|\sigma(x,\xi)| \geq C(1+|\xi|)^m, \quad |\xi| \geq R.$$

Then we say that σ is elliptic or T_σ is elliptic

Another Type of Ellipticity

Definition: Let $\sigma \in S^{2m}$, $m \in \mathbb{R}$. Suppose that there exist positive constants C and R such that

$$\operatorname{Re}\sigma(x,\xi) \geq C(1+|\xi|)^m, \quad |\xi| \geq R.$$

Then we say that σ is strongly elliptic or T_σ is strongly elliptic

Remarks: 1. If σ is strongly elliptic, then σ is elliptic.

The converse is not true. Why? (Exercise)

2. Why strongly elliptic?

A similar argument holds when

$$\partial_x^\alpha \partial_\xi^\beta = \partial_x^\gamma \partial_{x_j} \partial_\xi^\beta, \quad |\gamma| = \ell-1 \text{ and } j = 1,2,\dots,n.$$

Theorem (The Gårding Inequality) Let $\sigma \in S^{2m}$, $m \in \mathbb{R}$, be strongly elliptic. Then we can find a positive constant C' and a real constant C_s for all $s \geq \frac{1}{2}$ such that

$$\operatorname{Re}(T_\sigma \varphi, \varphi) \geq C' \|\varphi\|^2_{m,2} - C_s \|\varphi\|^2_{m-s,2}, \quad \varphi \in \mathcal{S}.$$

For a proof, we need some preparation. First, we identify $\mathbb{C}$ with $\mathbb{R}^2$ via

$$\mathbb{C} \ni z = x + iy \leftrightarrow (x, y) \in \mathbb{R}^2.$$

Lemma: Let $F \in C^\infty(\mathbb{C})$. Then for all $\sigma \in S^0$, $F \circ \sigma \in S^0$.

Proof: We need to prove that for all multi-indices α and β, there exists a positive constant $C_{\alpha\beta}$ such that

$$(*) \quad \left|\left(D_x^\alpha D_\xi^\beta (F \circ \sigma)\right)(x, \xi)\right| \leq C_{\alpha\beta} (1 + |\xi|)^{-|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

$(*)$ is true for all α and β with $|\alpha + \beta| = 0$, i.e., $\begin{cases} \alpha = 0, \\ \beta = 0. \end{cases}$

Indeed, there exists a positive constant C such that

$$|\sigma(x, \xi)| \leq C, \quad x, \xi \in \mathbb{R}^n.$$

Obviously,

$$F \circ \sigma \in C^\infty(\mathbb{R}^n).$$

Let $S = \{\sigma(x, \xi) : x, \xi \in \mathbb{R}^n\} \subset \mathbb{C}$. Then S is a bounded subset of $\mathbb{C}$. $\therefore$ $F \circ \sigma$ is a bounded function on $\mathbb{R}^n \times \mathbb{R}^n$.

Now, suppose that $(*)$ is true for all $F \in C^\infty(\mathbb{C})$, $\sigma \in S^0$ and α, β with $|\alpha + \beta| = l$. Let α and β be multi-indices with $|\alpha + \beta| = l + 1$. We first suppose that

$$\partial_x^\alpha \partial_\xi^\beta = \partial_x^\alpha \partial_\xi^\gamma \partial_{\xi_j}$$

for some γ with $|\gamma| = l$ and $j = 1, 2, \cdots, n$. By the Chain Rule,

$$\partial_x^\alpha \partial_\xi^\beta (F \circ \sigma) = \partial_x^\alpha \partial_\xi^\gamma \{(F_1 \circ \sigma)(\partial_{\xi_j} \sigma) + (F_2 \circ \sigma)(\partial_{\xi_j} \sigma)\},$$

where F_1 and F_2 are, respectively, the first partial derivatives of F with respect to the first and second variables.

Now,

$$\left|\left(\partial_x^\alpha \partial_\xi^\gamma \{(F_1 \circ \sigma)(\partial_{\xi_j} \sigma)\}\right)(x, \xi)\right|$$

$$\leq \sum_{\substack{\rho \leq \alpha \\ \delta \leq \gamma}} \binom{\alpha}{\rho}\binom{\gamma}{\delta} \left|\left(\partial_x^\rho \partial_\xi^\delta (F_1 \circ \sigma)\right)(x, \xi)\right| \left|\left(\partial_x^{\alpha-\rho} \partial_\xi^{\gamma-\delta}\right)(\partial_{\xi_j} \sigma)\right)(x, \xi)\right|$$

$$\leq \sum_{\substack{\rho \leq \alpha \\ \delta \leq \gamma}} \binom{\alpha}{\rho}\binom{\gamma}{\delta} C_{\rho\delta} (1 + |\xi|)^{-|\delta|} C_{\alpha\rho\gamma\delta j} (1 + |\xi|)^{-|\gamma| + |\delta| - 1}$$

$$= \underbrace{\sum_{\substack{\rho \leq \alpha \\ \delta \leq \gamma}} \binom{\alpha}{\rho}\binom{\gamma}{\delta} C_{\rho\delta} C_{\alpha\rho\gamma\delta j}}_{C_{\alpha\gamma j}} (1 + |\xi|)^{-(|\gamma|+1)}, \quad x, \xi \in \mathbb{R}^n.$$

Similarly, we can find a positive constant $C'_{\alpha\gamma j}$ such that

$$\left|\left(\partial_x^\alpha \partial_\xi^\gamma \{(F_2 \circ \sigma)(\partial_{\xi_j} \sigma)\}\right)(x, \xi)\right| \leq C'_{\alpha\gamma j} (1 + |\xi|)^{-(|\gamma|+1)}, \quad x, \xi \in \mathbb{R}^n.$$

$$\therefore \left|\left(\partial_x^\alpha \partial_\xi^\beta \{F \circ \sigma\}\right)(x, \xi)\right| \leq (C_{\alpha\gamma j} + C'_{\alpha\gamma j})(1 + |\xi|)^{-|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$