Lecture 23

Selected Exercises from the Textbook

20.3. Let f be a nonstant holomorphic function on a domain D. Then $|f|$ can attain a local minimum only at a zero of f in D. (Minimum Modulus Principle)

Proof: Suppose that $|f|$ attains a local minimum at $z_0 \in D$. If $f(z_0) = 0$, then we are done. If $f(z_0) \neq 0$, then there exists a positive number R such that

$$|f(z_0)| \leq |f(z)|, \quad |z - z_0| < R.$$

Then

$$\frac{1}{|f(z)|} \leq \frac{1}{|f(z_0)|}, \quad |z - z_0| < R.$$

$\therefore \frac{1}{|f|}$ attains a local maximum at z_0 in D. Since $\frac{1}{f}$ is holomorphic on $\{z \in \mathbb{C} : |z - z_0| < R\}$. By the Maximum Modulus Principle for Holomorphic Functions, $\frac{1}{f}$ is constant on $\{z \in \mathbb{C} : |z - z_0| < R\}$, i.e., f is constant on $\{z \in \mathbb{C} : |z - z_0| < R\}$. By the unique continuation property of holomorphic functions, f is a constant function on D. This is a contradiction.

An Application of

The Fundamental Theorem of Algebra: 20.4

Let $P(z)$ be a nonconstant polynomial of z in $\mathbb{C}$, i.e.,

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where $a_0, a_1, a_2, \ldots, a_n$ are complex constants with $a_n \neq 0$. Then $P(z)$ has a zero in $\mathbb{C}$.

Proof: Suppose that $P(z)$ has no zeros in $\mathbb{C}$. Then $\frac{1}{P}$ is holomorphic on $\mathbb{C}$. Then by the Maximum Modulus Principle for holomorphic functions, if $\frac{1}{|P|}$ has a local minimum at $z_0 \in \mathbb{C}$, then there exists a positive number R with

$$\left|\frac{1}{P(z)}\right| \geq \left|\frac{1}{P(z_0)}\right|, \quad |z - z_0| < R,$$

or $\quad |P(z)| \leq |P(z_0)|, \quad |z - z_0| < R.$

$\therefore$ $|P(z)|$ attains its local maximum at $z_0 \in \mathbb{C}$.

By the Maximum Modulus Principle for Holomorphic Functions on $\mathbb{C}$, this is a contradiction.

Exercise 20.5

Let $f: \partial\mathbb{D} \to \mathbb{C}$ be a continuous function. Is there a holomorphic function u on $\mathbb{D}$ such that

$$u(re^{i\theta}) \to f(e^{i\theta})$$

uniformly with respect to θ in $[0, 2\pi]$ as $r \to 1-$?

Solution

Let $\partial\mathbb{D} \to \mathbb{C}$ be defined by
$$f(z) = \bar{z}, \quad z \in \mathbb{C}.$$
Let u be a holomorphic function on $\mathbb{D}$ defined by
$$u(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}, \quad 0 < r < 1, \ 0 \le \theta \le 2\pi.$$
Since $u(re^{i\theta}) \to f(e^{i\theta})$ uniformly with respect to θ in $[0, 2\pi]$ as $r \to 1-$, we have
$$\int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{in\theta}\, d\theta \longrightarrow \int_0^{2\pi} e^{-i\theta}\, d\theta = 0$$
as $r \to 1-$.

By uniform convergence,
$$a_0 2\pi + \sum_{n=1}^{\infty} a_n r^n \int_0^{2\pi} e^{in\theta}\, d\theta = 0$$
$\therefore$ $a_0 = 0$.
$$\therefore\ u(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n z^n = z \sum_{n=1}^{\infty} a_n z^{n-1}$$
$$= z \sum_{m=0}^{\infty} a_{m+1} z^m$$
So, $a_1 = 0$, and by repeating the process
$$a_2 = a_3 = \cdots = 0$$
$\therefore$ $u(z) = 0$,

and this is a contradiction.