Lecture 21

The Schwarz Problem on $\mathbb{D}$: Let $f: \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. Find all holomorphic functions u on $\mathbb{D}$ such that

$$\operatorname{Re} u(re^{i\theta}) \to f(e^{i\theta})$$

as $r \to 1-$ uniformly with respect to θ on $[0, 2\pi]$.

Theorem: Let $f: \partial\mathbb{D} \to \mathbb{R}$ be continuous. Then all solutions u of the Schwarz problem on $\mathbb{D}$ are given by

$$u(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\omega)\frac{\omega+z}{\omega-z}\frac{d\omega}{\omega} + ic, \quad z \in \mathbb{D},$$

where c is a real constant.

Proof: u is holomorphic on $\mathbb{D}$?

Let $z \in \mathbb{D}$. Then for all $\omega \in \partial\mathbb{D}$,

$$\frac{\omega+z}{\omega-z} = \frac{\omega}{\omega-z} + \frac{z}{\omega-z} = \frac{1}{1-\frac{z}{\omega}} + \frac{z}{\omega}\frac{1}{1-\frac{z}{\omega}}$$

$$= \sum_{n=0}^{\infty}\left(\frac{z}{\omega}\right)^n + \frac{z}{\omega}\sum_{n=0}^{\infty}\left(\frac{z}{\omega}\right)^n$$

$$= 1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{\omega}\right)^n,$$

where the convergence is with respect to $\omega \in \partial\mathbb{D}$. To see this,

$$\sum_{n=1}^{\infty}\left|\frac{z}{\omega}\right|^n \le \sum_{n=1}^{\infty}|z|^n < \infty \text{ for all } \omega \in \partial\mathbb{D}.$$

$\therefore$ $$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\omega)\frac{\omega+z}{\omega-z}\frac{d\omega}{\omega} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\omega)\frac{d\omega}{\omega} + \frac{1}{\pi i}\left(\sum_{n=1}^{\infty}\frac{f(\omega)}{\omega^n}\frac{d\omega}{\omega}\right)z^n$$

for all $z \in \mathbb{D}$. $\therefore$ u is holomorphic on $\mathbb{D}$.

• $\operatorname{Re} u(re^{i\theta}) \to f(e^{i\theta})$ as $r \to 1-$ uniformly with respect to θ on $[0, 2\pi]$.

Write $z = re^{i\theta}$, $0<r<1$, $0 \le \theta \le 2\pi$, $\omega = e^{i\phi}$, $0 \le \phi \le 2\pi$. Let $F(\theta) = f(e^{i\theta})$, $0 \le \theta \le 2\pi$.

Then

$$u(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} F(\phi)\,\frac{e^{i\phi}+re^{i\theta}}{e^{i\phi}-re^{i\theta}}\, i\,d\phi + ic.$$

But

$$\frac{e^{i\phi}+re^{i\theta}}{e^{i\phi}-re^{i\theta}} = \frac{e^{i\phi}+re^{i\theta}}{e^{i\phi}-re^{i\theta}}\,\frac{e^{-i\phi}-re^{-i\theta}}{e^{-i\phi}-re^{-i\theta}}$$

$$= \frac{1-re^{-i(\theta-\phi)}+re^{i(\theta-\phi)}-r^2}{1-re^{-i(\theta-\phi)}-re^{i(\theta-\phi)}+r^2}$$

$$= \frac{1-r^2+2i\,\operatorname{Im}(\sin(\theta-\phi))}{1-2r\cos(\theta-\phi)+r^2}$$

$\therefore$ $\operatorname{Re} u(re^{i\theta})$

$$= \frac{1}{2\pi}\int_0^{2\pi} F(\phi)\,\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\,d\phi.$$

So $\operatorname{Re} u(re^{i\theta}) - f(e^{i\theta})$

$$= \frac{1}{2\pi}\int_0^{2\pi}\big(F(\phi)-F(\theta)\big)\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\,d\phi.$$

Now, F is continuous on $[0, 2\pi]$, $\therefore$ F is uniformly continuous on $[0, 2\pi]$. So for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all θ and ϕ in $[0, 2\pi]$ with $|\theta - \phi| < \delta$,

$$|F(\theta) - F(\phi)| < \frac{\varepsilon}{3}.$$

$$|\operatorname{Re} u(re^{i\theta}) - f(e^{i\theta})|$$

$$= \frac{1}{2\pi}\left|\left(\int_{|\phi-\theta|<\delta} + \int_{\delta \le |\phi-\theta| \le 2\pi-\delta} + \int_{2\pi-\delta\le|\phi-\theta|\le 2\pi}\right) | \cdots | \, d\phi\right|$$

$$< \frac{2\delta}{3} + \frac{1}{2\pi}\int_{\delta\le|\phi-\theta|\le 2\pi-\delta} |F(\phi)-F(\theta)| \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, d\phi$$

Now

$$\frac{1}{1-2r\cos(\theta-\phi)+r^2} \le \frac{1}{1-2r\cos\delta + r^2},$$

and this has an absolute minimum at $r = \cos\delta$.

$$\therefore \frac{1}{1-2r\cos(\theta-\phi)+r^2} \le \frac{1}{1-\cos^2\delta}$$

Let $M = \max_{0\le\phi\le 2\pi} |F(\phi)|$. Then

$$\frac{1}{2\pi}\int_{\delta\le|\phi-\theta|\le 2\pi-\delta} |F(\phi)-F(\theta)| \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, d\phi$$

$$\le \frac{M}{\pi}\int_{\delta\le|\phi-\theta|\le 2\pi-\delta} \frac{1-r^2}{1-\cos^2\delta}\, d\phi \le 2M\frac{1-r^2}{1-\cos^2\delta}$$

$$< \frac{\varepsilon}{3}$$

whenever $0<|r-1|<\delta_0$ for some $\delta_0>0$.

$\therefore$ $0<|r-1|<\delta_0 \Rightarrow |\operatorname{Re} u(re^{i\theta}) - f(e^{i\theta})| < \underbrace{\frac{2\varepsilon}{3} + \frac{\varepsilon}{3}}_{=\varepsilon}$

$\therefore$ $\operatorname{Re} u(re^{i\theta}) \to f(e^{i\theta})$ as $r \to 1-$ uniformly with respect to θ in $[0, 2\pi]$.