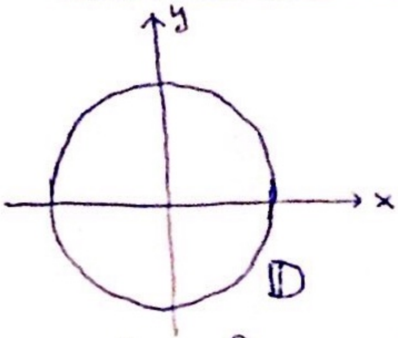
Lecture 20

Complex Analysis in PDE: The Schwarz Problem on $\mathbb{D}$



Let f be a continuous real-valued function on $\partial\mathbb{D}$. Find all holomorphic functions u on $\mathbb{D}$ such that

$$\operatorname{Re} u(re^{i\theta}) \to f(e^{i\theta})$$

uniformly with respect to θ in $[0, 2\pi]$ as $r \to 1-$.

Motivation 1: Let u be a holomorphic function on $\overline{\mathbb{D}}$ such that $\operatorname{Re} u = f$ on $\partial\mathbb{D}$. Then

$$2f(\omega) = u(\omega) + \overline{u(\omega)}, \quad \omega \in \partial\mathbb{D}.$$

So,

$$2f(\omega) = \sum_{m=0}^{\infty} a_m \omega^m + \sum_{m=0}^{\infty} \overline{a_m \omega^m}, \quad \omega \in \partial\mathbb{D}.$$

The convergence of each series is uniform on $\partial\mathbb{D}$. Then for $n = 0, 1, 2, \ldots$,

$$2\int_{\partial\mathbb{D}} \frac{f(\omega)}{\omega^n}\frac{d\omega}{\omega} = \sum_{m=0}^{\infty} a_m \int_{\partial\mathbb{D}} \frac{\omega^m}{\omega_n}\frac{d\omega}{\omega} + \sum_{m=0}^{\infty} \bar{a}_m \int_{\partial\mathbb{D}} \frac{\overline{\omega^m}}{\omega_n}\frac{d\omega}{\omega}.$$

But

$$\int_{\partial\mathbb{D}} \frac{\omega^m}{\omega^n}\frac{d\omega}{\omega} = \int_0^{2\pi} e^{i(m-n)\theta}\frac{ie^{i\theta}\,d\theta}{e^{i\theta}} = \int_0^{2\pi} e^{i(m-n)\theta}\, i\,d\theta$$

$$= \begin{cases} 2\pi i, & m = n, \\ 0, & m \neq n. \end{cases}$$

Also,

$$\int_{\partial\mathbb{D}} \frac{\overline{\omega^m}}{\omega^n}\frac{d\omega}{\omega} = \int_0^{2\pi} e^{-i(m+n)\theta}\, i\,d\theta = \begin{cases} 2\pi i, & m = n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\therefore\ 2\int_{\partial\mathbb{D}} \frac{f(\omega)}{\omega^n}\frac{d\omega}{\omega} = \begin{cases} (a_0 + \bar{a}_0)(2\pi i), & n = 0 \\ a_n, & n \geq 1 \end{cases}$$

$$\therefore \frac{1}{\pi i}\int_{\partial D} \frac{f(\omega)}{\omega^n}\frac{d\omega}{\omega} = \begin{cases} a_0+\overline{a_0}, & n=0,\\ a_n, & n\geq 1.\end{cases}$$

$$\therefore u(z)=\sum_{n=0}^{\infty} a_n z^n = \frac{1}{\pi i}\sum_{n=0}^{\infty}\int_{\partial D} f(\omega)\left(\frac{z}{\omega}\right)^n \frac{d\omega}{\omega} - \overline{a_0}$$

But $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{\omega}\right)^n = \frac{1}{1-\frac{z}{\omega}} = \frac{\omega}{\omega-z}$. $\therefore$

$$u(z) = \frac{1}{\pi i}\int_{\partial D}\frac{\cancel{\omega}}{\omega-z} f(w)\frac{dw}{\cancel{\omega}} - \overline{a_0}$$

But $\displaystyle \operatorname{Re} a_0 = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\omega)}{\omega}\,d\omega.$ $\therefore$

$$u(z) = \frac{1}{\pi i}\int_{\partial D}\frac{f(\omega)}{\omega-z}\,d\omega - \operatorname{Re} a_0 - i\operatorname{Im} a_0$$

$$= \frac{1}{\pi i}\int_{\partial D}\frac{f(w)}{\omega-z}\,dw - \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{\omega}\,dw - i\operatorname{Im} a_0$$

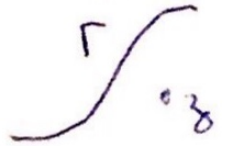
$$= \frac{1}{\pi i}\int_{\partial D} f(\omega)\frac{2\omega-(\omega-z)}{2(\omega-z)\omega}\,dw - i\operatorname{Im} a_0$$

$$= \frac{1}{\pi i}\int_{\partial D} f(w)\frac{w+z}{w-z}\frac{dw}{w} - i\operatorname{Im} a_0.$$

We also need Theorem 10.5

Theorem 10.5: Let Γ be a contour in $\mathbb{C}$. Let $w=g(z)$ be a continuous function on Γ. Let G be the function on $\mathbb{C}-\Gamma$ defined by

$$G(z) = \int_{\Gamma}\frac{g(S)}{S-z}\,dS,\quad z\in\mathbb{C}-\Gamma.$$



Then G is holomorphic on $\mathbb{C}-\Gamma$ and

$$G'(z)=\int_\Gamma \frac{g(\zeta)}{(\zeta-z)^2}\,d\zeta,\quad z\in\mathbb{C}-\Gamma.$$

Proof: We need to prove that

$$\lim_{h\to 0}\frac{G(z+h)-G(z)}{h}=\int_\Gamma\frac{g(\zeta)}{(\zeta-z)^2}\,d\zeta,\quad z\in\mathbb{C}-\Gamma.$$

But

$$\frac{G(z+h)-G(z)}{h}=\frac{1}{h}\int_\Gamma\left\{\frac{1}{\zeta-(z+h)}-\frac{1}{\zeta-z}\right\}g(\zeta)\,d\zeta$$

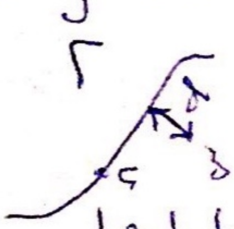
$$=\frac{1}{\cancel{h}}\int_\Gamma\frac{\cancel{h}\,g(\zeta)}{(\zeta-(z+h))(\zeta-z)}\,d\zeta$$

$$\therefore\ \frac{G(z+h)-G(z)}{h}-\int_\Gamma\frac{g(\zeta)}{(\zeta-z)^2}\,d\zeta$$

$$=\int_\Gamma\frac{g(\zeta)}{(\zeta-z-h)(\zeta-z)}\,d\zeta-\int_\Gamma\frac{g(\zeta)}{(\zeta-z)^2}\,d\zeta$$

$$=\int_\Gamma g(\zeta)\frac{\zeta-z-(\zeta-z-h)}{(\zeta-z-h)(\zeta-z)^2}\,d\zeta=h\int_\Gamma\frac{g(\zeta)}{(\zeta-z-h)(\zeta-z)^2}\,d\zeta$$

$$=J_h(z).$$



Now $|\zeta-z-h|\geq|\zeta-z|-|h|\geq d-\frac{d}{2}=\frac{d}{2}$

$|\zeta-z|^2\geq d^2$

$|g(\zeta)|\leq M$ for some $M>0$, $\zeta\in\Gamma$

Let h be $\ni |h|<\frac{d}{2}$

$$\therefore\ |J_h(z)|\leq|h|\left(M\Big/\left(\tfrac{d}{2}\right)d^2\right)|\Gamma|\to 0 \text{ as } h\to 0,$$

where $|\Gamma|=$ length of Γ. $\therefore\ J_h(z)\to 0$ as $h\to 0$.

$$\therefore\ G'(z)=\int_\Gamma\frac{g(\zeta)}{(\zeta-z)^2}\,d\zeta,\quad z\in\mathbb{C}-\Gamma.$$