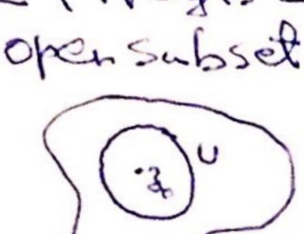
Harmonic Functions in Detail

Theorem (Weyl's Lemma) Let u be a harmonic function on an open subset $G \subseteq \mathbb{C}$. Then $u \in C^\infty(G)$.

Proof:



Let $z_0 \in G$. Let U be an open disk centred at z_0 and $U \subseteq G$. Then u is harmonic on U. Let v be a harmonic conjugate of u on U. Then $f = u + iv$ is holomorphic on U. $\therefore$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

$\therefore u \in C^1(U)$

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} - i\frac{\partial^2 u}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial y \partial x} - i\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} + i\frac{\partial^2 u}{\partial y^2}.$$

$\therefore$ $u \in C^2(U), \ldots$

$\therefore$ $u \in C^\infty(U)$. But z_0 is an arbitrary point in U, $\therefore u \in C^\infty(G)$.

Theorem (Mean Value Property)
Let u be a harmonic function on a simply connected domain D. Then for all circles

$$\{z \in D : |z - z_0| = r\} \subseteq D,$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta.$$

Proof: Let v be a harmonic conjugate of u on D.
Then $f = u + iv$ is holomorphic on D. By the mean value property of holomorphic functions on D,

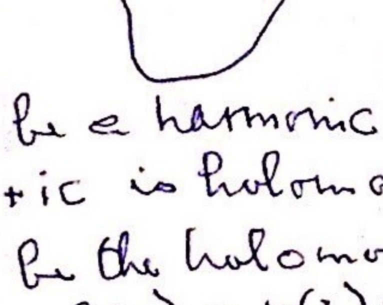
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

$$\therefore\ u(z_0) = (\operatorname{Re} f)(z_0) = \frac{1}{2\pi}\int_0^{2\pi} (\operatorname{Re} f)(z_0 + re^{i\theta})\,d\theta.$$

$$\therefore\ u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta.$$

Theorem (Unique Continuation Property)
Let u be a harmonic function on a simply connected domain D such that $u = 0$ on a nonempty and open subset U of D. Then $u = 0$ on D.

Proof:



Let $z_0 \in D$. Let U be an open disk centered at z_0 such that $u = 0$ on U.

Let v be a harmonic conjugate of u on D. Then $u + iv + ic$ is holomorphic on D for every constant c. Let f be the holomorphic function on D defined by

$$f(z) = u(z) + iv(z) - iv(z_0),\quad z \in D.$$

Then $f(z_0) = u(z_0) = 0$. Since

$\operatorname{Re} f = u = 0$ on U, $\therefore$

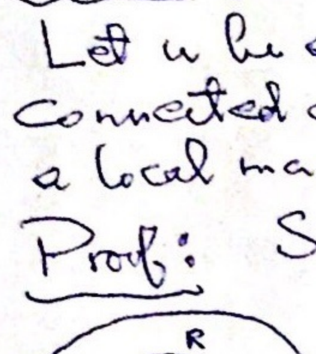
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0 \text{ on } U$$

Since f' is holomorphic on D, by UCP for holomorphic functions, $f'(z) = 0$ for all $z \in D$. $\therefore f(z) = C$, where C is a constant on D. But $f(z_0) = 0$. $\therefore C = 0$.

$\therefore f(z) = 0$ for all $z \in D$.

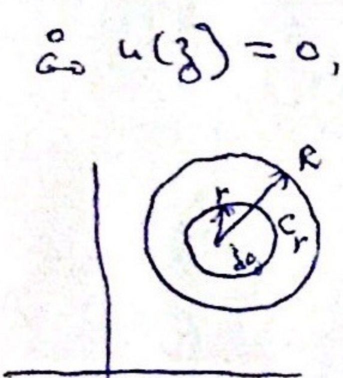
<u>Theorem</u> (Maximum Modulus Principle)
Let u be a nonconstant harmonic function on a simply connected domain D. Then $|u(z)|$ cannot attain a local maximum in D.

<u>Proof:</u> Suppose that $|u(z)|$ attains a local maximum at $z_0 \in D$. Then there exists a positive number R such that



$$\begin{cases} \{z \in D : |z - z_0| < R\} \subset D, \\ |u(z_0)| \geq |u(z)|, \ |z - z_0| < R. \end{cases}$$

If $u(z_0) = 0$, then
$$|u(z)| = 0, \ |z - z_0| < R.$$
$\therefore u(z) = 0$, $z \in D$. Suppose $u(z_0) \neq 0$. Let
$$\lambda = \frac{|u(z_0)|}{u(z_0)}.$$



Then by the MVP of harmonic functions,
$$|u(z_0)| = \lambda u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \lambda u(z_0 + re^{i\theta})\, d\theta.$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} \left(|u(z_0)| - \lambda u(z_0 + re^{i\theta})\right) d\theta = 0.$$

But $|u(z_0)| \geq |\lambda u(z_0 + re^{i\theta})| = |u(z)|$ on C_r.

$\therefore$ on C_r,

$$u(z) = \frac{|u(z_0)|}{\lambda}$$

Let $r \to 0$,

$$\therefore u(z) = \frac{|u(z_0)|}{\lambda}, \qquad |z - z_0| < R.$$

$\therefore u(z) =$ Constant on D by the UCP.

Theorem (Another Version of MMP for Harmonic Functions)
Let u be a harmonic function on a simply connected domain D. Let K be a compact subset of D. Then $\max_{z \in K} |u(z)|$ is attained at some boundary point of K.