Lecture 18

Harmonic Functions

Let G be an open set in $\mathbb{C}$. Let $u: G \to \mathbb{R}$ be such that $u \in C^2(G)$ and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for all $z = (x, y)$ in G.

Then we call u a harmonic function on G.

Theorem: Let f be a holomorphic function on G. Let $f = u + iv$, where u and v are the real part and the imaginary part of f. Then u and v are harmonic on G.

Proof: (For u only)

f is holomorphic on G

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$

by Cauchy-Riemann equations. $\therefore u \in C^1(G)$.

f' is holomorphic on G

$$\Rightarrow f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} + i\frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial y\,\partial x} - i\frac{\partial^2 v}{\partial y\,\partial x} = \frac{\partial^2 v}{\partial x\,\partial y} - i\frac{\partial^2 u}{\partial x\,\partial y}$$

$\Rightarrow \cdots$

$\therefore u \in C^\infty(G)$. Also,

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x\,\partial y}, \\[2mm] \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial y\,\partial x}. \end{cases}$$

Since $v \in C^2(G)$, we get by Clairaut's theorem,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}.$$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ on G. $\therefore$ u is harmonic on G.

<u>Theorem</u>: Let u be a harmonic function on a <u>simply connected domain</u> D. Then there exists a harmonic function v on D such that $u + iv$ is holomorphic on D. If w is another harmonic function on D such that $u + iw$ is holomorphic on D, then $v - w$ is a constant function on D.

<u>Remarks</u>: The function v is called a harmonic conjugate of u. Two harmonic conjugates of the same harmonic function differ by a constant.

<u>Proof of Theorem</u>: For all $z = (x, y) \in D$, let

$$f(z) = U(x, y) + i V(x, y),$$

where

$$\begin{cases} U(x,y) = \dfrac{\partial u}{\partial x}, \\ V(x,y) = -\dfrac{\partial u}{\partial y}. \end{cases}$$

$$\therefore \begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial V}{\partial y}, \\ \dfrac{\partial U}{\partial y} = \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y} = -\dfrac{\partial V}{\partial x}. \end{cases}$$

By Cauchy-Riemann equations, f is holomorphic on D. By Cauchy's integral theorem,

$$\int_\Gamma f(z)\,dz = 0$$

for all closed contours Γ in D. $\therefore$ f has an antiderivative $F = \mu + i\nu$ on D. $\therefore$

$$F'(z) = \frac{\partial \mu}{\partial x} + i\frac{\partial \nu}{\partial x} = \frac{\partial \nu}{\partial y} - i\frac{\partial \mu}{\partial y}, \quad z \in D.$$

$$\therefore \begin{cases} \dfrac{\partial \mu}{\partial x} = \dfrac{\partial u}{\partial x}, \\[2ex] \dfrac{\partial \mu}{\partial y} = \dfrac{\partial u}{\partial y}. \end{cases} \qquad \therefore \mu - u = C, \text{ a constant (real) on } D.$$

Let $G : D \to \mathbb{C}$ be such that $G = F - C$. Then G is holomorphic on D and

$$\operatorname{Re} G = (\operatorname{Re} F) - C = \mu - C = u$$

Taking ν to be v, then

$$u + iv = (\mu - C) + iv$$

is holomorphic on D.

Now, let v and w be harmonic functions on D such that $u + iv$ and $u + iw$ are holomorphic on D.

Then by Cauchy-Riemann equations,

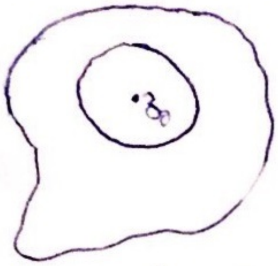
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{\partial w}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = -\dfrac{\partial w}{\partial x} \end{cases}$$

$\therefore$ $v - w = \text{Constant on } D$

Theorem (Weyl's lemma)

Let G be an open subset of $\mathbb{C}$. Let $u : G \to \mathbb{R}$ be harmonic. Then $u \in C^\infty(G)$.

Proof:



Let $z_0 \in G$. Then there exists an open disk U centered at z_0 such that u is harmonic on U.

Let v be a harmonic conjugate of u on U. Then $f = u + iv$ is holomorphic on U. $\therefore$ for all $z = (x, y) \in U$,

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y},$$

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} - i \frac{\partial^2 u}{\partial x \partial y}$$

$$= \frac{\partial^2 u}{\partial y^2} + i \frac{\partial^2 u}{\partial y^2},$$

$\vdots$

$\therefore$ All partial derivatives of u with respect to x and y exist. $\therefore$ $u \in C^\infty(G)$.