# Lecture 17

**Theorem:** $\mathrm{Aut}(\mathbb{H}) = \{ f_A : A \in SL(2,\mathbb{R}) \}$.

**Remarks:** $\mathrm{Aut}(\mathbb{H})$ and $\{ f_A : A \in SL(2,\mathbb{R}) \}$ are not isomorphic groups. For if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, then

$$f_A(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{H},$$

$$f_{-A}(z) = \frac{-az-b}{-cz-d} = \frac{az+b}{cz+d} = f_A(z), \quad z \in \mathbb{H}.$$

**Theorem:** The Center $Z$ of $SL(2,\mathbb{R})$ is given by

$$Z = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

**Proof:** Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Z$. Then for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$a\alpha + b\gamma = a\alpha + c\beta \quad (1)$$
$$a\beta + b\delta = b\alpha + d\beta \quad (2)$$
$$c\alpha + d\gamma = a\gamma + c\delta \quad (3)$$
$$c\beta + d\delta = b\gamma + d\delta \quad (4)$$

By (1), $b\gamma = c\beta$. Let $c = 0$. Then $b\gamma = 0$. If $b \neq 0$, then $\gamma = 0$.

$\therefore c\beta = 0 \Rightarrow \beta = 0$ if we choose $c \neq 0$.

By (3), $\alpha = \delta$ if we choose $c \neq 0$.

$\therefore \alpha\delta - \beta\gamma = 1 \Rightarrow \alpha = \delta = 1$ or $\alpha = \delta = -1$

$$\therefore Z = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

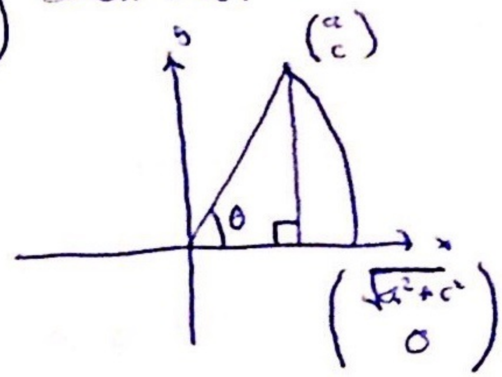
Since $Z$ is a normal subgroup of $SL(2,\mathbb{R})$. $\therefore$

$$SL(2,\mathbb{R})/Z = Z\backslash SL(2,\mathbb{R}) = \mathrm{Aut}(\mathbb{H}).$$

## The Iwasawa Decomposition of $SL(2,\mathbb{R})$

Let $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{R})$. Then there exists a rotation matrix $k=\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$ such that the column $\begin{pmatrix} a \\ c\end{pmatrix}$ is transformed into

$$\begin{pmatrix}\sqrt{a^2+c^2} \\ 0\end{pmatrix}$$




Now

$$kg=\begin{pmatrix}\sqrt{a^2+c^2} & x \\ 0 & \dfrac{1}{\sqrt{a^2+c^2}}\end{pmatrix}$$

$$\cos\theta = \frac{a}{\sqrt{a^2+c^2}},\quad \sin\theta = \frac{c}{\sqrt{a^2+c^2}}.$$

$$\therefore kg = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} \Longrightarrow x = b\cos\theta + d\sin\theta = \frac{ab+cd}{\sqrt{a^2+c^2}}$$

$$\therefore kg = \begin{pmatrix}\sqrt{a^2+c^2} & \dfrac{ab+cd}{\sqrt{a^2+c^2}} \\ 0 & \dfrac{1}{\sqrt{a^2+c^2}}\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{a^2+c^2} & 0 \\ 0 & \dfrac{1}{\sqrt{a^2+c^2}}\end{pmatrix}\begin{pmatrix} 1 & \dfrac{ab+cd}{a^2+c^2} \\ 0 & 1\end{pmatrix}$$

So $g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{a^2+c^2} & 0 \\ 0 & \frac{1}{\sqrt{a^2+c^2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 1 \end{pmatrix}$.

<u>Theorem</u> (Iwasawa Decomposition)

$$SL(2,\mathbb{R}) = KAN,$$

where

$$K = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R}, \text{ Compact}$$

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \alpha > 0, \text{ abelian}$$

$$N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R}, \text{ nilpotent}$$

Here, $\alpha = \sqrt{a^2+c^2}$,

$$b = \frac{ab+cd}{a^2+c^2}$$

<u>A Glimpse into the Affine Group</u>

The group $K$ given by

$$K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

is called the <u>special orthogonal group</u>, denoted by $SO(2,\mathbb{R})$.

$$SO(2,\mathbb{R}) \backslash SL(2,\mathbb{R}) \overset{\text{identified}}{\simeq} AN \overset{\text{identified}}{\simeq} \{(b,a) : b \in \mathbb{R}, a > 0\} = \mathbb{H}.$$

(Exercise 22.7)

# Harmonic Functions



Definition Let $D$ be a domain in $\mathbb{C}$. Let $u: D \to \mathbb{R}$ be a $C^2$ function on $D$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on $D$. Then we call $u$ a harmonic function on $D$

Remark: We shall only consider simply connected domain $D$ unless specified otherwise.

Theorem: Let $f = u + iv$ be a holomorphic function on a domain $D$. Then $u$ and $v$ are harmonic on $D$.

Proof: By Cauchy-Riemann equations,

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{on } D.$$

$$\therefore \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y} \\[2ex] \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial y \partial x} \end{cases} \quad \text{on } D.$$

Since $v \in C^2(D)$, $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$. $\therefore$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on } D$$

$\therefore$ $u$ is harmonic on $D$

Similarly, $v$ is harmonic on $D$.