Lecture 16

To study Aut($\mathbb{H}$), let

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} ; ad-bc=1 \right\}.$$

It is a group with respect to matrix multiplication. We call it the <u>special linear group</u>.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. Then we define the fractional linear transformation $f_A : \mathbb{H} \to \mathbb{C}$ by

$$f_A(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{H}.$$

Observations:

1) $c=0$ or $d=0$, but not both.
2) $c=0 \Rightarrow f_A$ is holomorphic on $\mathbb{H}$.
3) $c \neq 0 \Rightarrow z \neq -\frac{d}{c} \Rightarrow f_A$ is holomorphic on $\mathbb{H}$.
4) f_A maps $\mathbb{H}$ into $\mathbb{H}$. To see this, let $z \in \mathbb{H}$. Then

$$\operatorname{Im}(f_A(z)) = \operatorname{Im}\frac{az+b}{cz+d} = \operatorname{Im}\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}$$

$$= \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2} > 0.$$

$\therefore f_A(z) \in \mathbb{H}$.

5) $f_A \circ f_B = f_{AB}$ for all $A, B \in SL(2,\mathbb{R})$. (Exercise 22.4)
6) For all $A \in SL(2,\mathbb{R})$,

$$f_A \circ f_{A^{-1}} = f_{AA^{-1}} = f_I = I.$$

7) For all $A \in SL(2,\mathbb{R})$, $f_A \in \operatorname{Aut}(\mathbb{H})$.

<u>Theorem</u>: $SL(2,\mathbb{R})$ acts transitively on $\mathbb{H}$. Precisely, §6.2
for all z and ω in $\mathbb{H}$, there exists a matrix $A \in SL(2,\mathbb{R})$ such that

$$f_A(z) = \omega.$$

<u>Proof</u>: It is enough to prove that for all $z \in \mathbb{H}$, there exists a matrix A in $SL(2,\mathbb{R})$ such that

$$f_A(z) = i \quad (*)$$

For then, if A and B are matrices in $SL(2,\mathbb{R})$ such that

$$f_A(z) = i$$

and

$$f_B(\omega) = i,$$

then

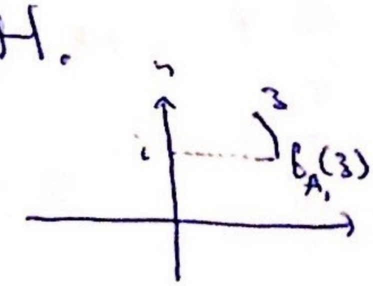
$$f_{B^{-1}A}(z) = f_{B^{-1}}(f_A(z)) = f_{B^{-1}}(i) = f_B^{-1}(i) = \omega.$$

To prove $(*)$, recall that

$$\operatorname{Im}(f_A(z)) = \frac{\operatorname{Im} z}{|cz+d|^2}, \quad z \in \mathbb{H}.$$

Let $d = 0$. Then

$$\operatorname{Im}(f_A(z)) = \frac{\operatorname{Im} z}{|cz|^2}, \quad z \in \mathbb{H}.$$



Choose $c \in \mathbb{R}$ such that

$$\operatorname{Im}(f_A(z)) = 1$$

and we let

$$A_1 = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix} \in SL(2,\mathbb{R}).$$

Let

$$A_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{R}), \text{ where } b = -\operatorname{Re}(f_{A_1}(z))$$

Let $A = A_2 A_1$. Then

$$f_A(z) = f_{A_2A_1}(z) = f_{A_2}(f_{A_1}(z)) = i.$$

A New Look at Rotations

Lemma: Let

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SL(2,\mathbb{R}), \quad \theta \in \mathbb{R}.$$

Then

$$F \circ f_{A_\theta} \circ F^{-1} = r_{-2\theta},$$

where $F : \mathbb{H} \to \mathbb{D}$ is the Cayley transform.

This is Lemma 22.3 in the book.

Theorem $\mathrm{Aut}(\mathbb{H}) = \{ f_A : A \in SL(2,\mathbb{R}) \}$

Proof: We have already proved that $\supseteq$ is true. Now, let $f \in \mathrm{Aut}(\mathbb{H})$. Let $\beta \in \mathbb{H}$ be such that

$$f(\beta) = i.$$

Let $B \in SL(2,\mathbb{R})$ be such that $f_B(i) = \beta$. Let

$$g = f \circ f_B.$$

Then $g(i) = f(f_B(i)) = i$. Note that $F \circ g \circ F^{-1}$ is holomorphic from $\mathbb{D}$ into $\mathbb{D}$ with

$$(F \circ g \circ F^{-1})(0) = 0.$$

$\therefore$ $F \circ g \circ F^{-1} = \cancel{A_\theta}$ $r_{-2\theta}$ for some $\theta \in \mathbb{R}$. $\therefore$

$$F \circ g \circ F^{-1} = F \circ f_{A_\theta} \circ F^{-1}.$$

$\therefore\quad g = f_{A_\theta}$, i.e.,

$$f \circ f_B = f_{A_\theta}.$$

$\therefore\quad f = f_{A_\theta} \circ f_{B^{-1}} = f_{A_\theta B^{-1}}.$

$\therefore\quad f \in \{ f_A : A \in SL(2, \mathbb{R}) \}.$