Lecture 14

$\mathrm{Aut}(\mathbb{D})$

Automorphisms of $\mathbb{D}$

1) Let $\theta \in \mathbb{R}$. Let $r_\theta : \mathbb{C} \to \mathbb{C}$ be defined by
$$r_\theta(z) = e^{i\theta} z, \; z \in \mathbb{C}.$$
Then $r_\theta \in \mathrm{Aut}(\mathbb{D})$ with inverse $r_{-\theta}$.

2) Let $\alpha \in \mathbb{C}$ be such that $|\alpha| < 1$. Then we define $\psi_\alpha : \mathbb{D} \to \mathbb{C}$ by
$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}, \quad z \in \mathbb{D}.$$
We call ψ_α a <u>Möbius transformation</u>.

Observations about ψ_α

1) ψ_α is holomorphic on $\mathbb{D}$.

2) ψ_α maps $\mathbb{D}$ into $\mathbb{D}$.

Proof: For all z with $|z| \in [1, \frac{1}{|\alpha|})$,
$$|\bar{\alpha} z| = |\alpha||z| < 1 \Rightarrow 1 - \bar{\alpha} z \neq 0.$$
$\therefore$ ψ_α is holomorphic on a neighborhood of $\overline{\mathbb{D}}$. Now, for $|z| = 1$, we write $z = e^{i\theta}$. We have
$$\left| \frac{\alpha - z}{1 - \bar{\alpha} z} \right| = \left| \frac{\alpha - e^{i\theta}}{1 - \bar{\alpha} e^{i\theta}} \right| = \left| e^{-i\theta} \frac{\alpha - e^{i\theta}}{e^{-i\theta} - \bar{\alpha}} \right|$$
$$= \left| e^{-i\theta} \frac{\omega}{\bar{\omega}} \right|, \text{ where } \omega = \alpha - e^{i\theta}.$$
$$\therefore \left| \psi_\alpha(z) \right| = \left| \frac{\alpha - z}{1 - \bar{\alpha} z} \right| = 1, \text{ for all } z \in \partial \mathbb{D}.$$
By the Maximum Modulus Principle,
$$|\psi_\alpha(z)| < 1, \; z \in \mathbb{D}.$$

3) For all $z \in \mathbb{D}$,

$$(\psi_\alpha \circ \psi_\alpha)(z) = \psi_\alpha(\psi_\alpha(z)) = \frac{\alpha - \psi_\alpha(z)}{1 - \bar{\alpha}\psi_\alpha(z)}$$

$$= \frac{\alpha - \frac{\alpha - z}{1-\bar{\alpha}z}}{1 - \bar{\alpha}\frac{\alpha - z}{1-\bar{\alpha}z}} = \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \bar{\alpha}z - |\alpha|^2 + \bar{\alpha}z}$$

$$= \frac{(1-|\alpha|^2)z}{1-|\alpha|^2} = z.$$

$\therefore \psi_\alpha \circ \psi_\alpha = I$.

This proves that $\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$ for all α with $|\alpha|<1$.

Two Useful Identities:

1) $\psi_\alpha(0) = \dfrac{\alpha - 0}{1 - \bar{\alpha}0} = \alpha$,

2) $\psi_\alpha(\alpha) = \dfrac{\alpha - \alpha}{1 - \bar{\alpha}\alpha} = 0$.

Theorem $\mathrm{Aut}(\mathbb{D}) = \{e^{i\theta}\psi_\alpha : \theta \in \mathbb{R}, \alpha \in \mathbb{D}\}$.

Proof: Let $f \in \mathrm{Aut}(\mathbb{D})$. Let $\alpha \in \mathbb{D}$ be such that $f(\alpha) = 0$.
Let $g = f \circ \psi_\alpha$. Then g is holomorphic on $\mathbb{D}$,

$$g(0) = f(\psi_\alpha(0)) = f(\alpha) = 0.$$

$\therefore |g(z)| \le |z|$, $z \in \mathbb{D}$ by Schwarz's lemma. Now, $g^{-1} = \psi_\alpha \circ f^{-1}$ is holomorphic on $\mathbb{D}$. Also,

$$g^{-1}(0) = \psi_\alpha(f^{-1}(0)) = \psi_\alpha(\alpha) = 0.$$

By Schwarz's lemma again,

$$|g^{-1}(w)| \le |w|, \quad w \in \mathbb{D}.$$

For all $z \in \mathbb{D}$, let $\omega = g(z)$. Then

$$|z| \leq |g(z)|, \; z \in \mathbb{D}.$$

$\therefore$ $|g(z)| = |z|$, $z \in \mathbb{D}$. By Schwarz's lemma again, $\therefore$ g is a rotation. $\therefore$ there exists a real number θ such that

$$g(z) = e^{i\theta} z, \; z \in \mathbb{D}.$$

$\therefore$ $g(\psi_\alpha(z)) = e^{i\theta}\psi_\alpha(z)$, $z \in \mathbb{D}$.

$\therefore$ $f(z) = e^{i\theta}\psi_\alpha(z)$, $z \in \mathbb{D}$. $\therefore$ $f = e^{i\theta}\psi_\alpha$.

<u>Corollary</u>: Let $f \in \mathrm{Aut}(\mathbb{D})$ be such that $f(0) = 0$. Then f is a rotation.

<u>Proof</u>: Let $f \in \mathrm{Aut}(\mathbb{D})$. Then $f(z) = e^{i\theta}\psi_\alpha(z)$, $z \in \mathbb{D}$.

$$f(0) = e^{i\theta}\psi_\alpha(0) = \alpha e^{i\theta} \Rightarrow \alpha = 0$$

$$\therefore f(z) = e^{i\theta}\psi_0(z) = e^{i\theta}\frac{0 - z}{1 - 0z} = -e^{i\theta}z \Rightarrow f \text{ is a rotation}$$

<u>Theorem</u>: $\mathrm{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$, i.e., for all α and β in $\mathbb{D}$ there exists an element $f \in \mathrm{Aut}(\mathbb{D})$ such that $\beta = f(\alpha)$.

<u>Proof</u>: Let $f = \psi_\beta \circ \psi_\alpha$. Then

$$f(\alpha) = \psi_\beta(\psi_\alpha(\alpha)) = \psi_\beta(0) = \beta.$$

Meaning of Transitive Group Actions

Let G be a group with binary operation $\circ$. Let X be a set, an arbitrary and abstract set. The <u>group action of G on X</u> is a mapping

$$G \times X \ni (g, x) \longmapsto gx \in X$$

such that

$$(g \circ h, x) \longmapsto g(hx)$$

and

$$(e, x) \longmapsto x$$

for all $g, h \in G$, e is the identity element in G and for all $x \in X$. For all $x, y \in X$, we say that $x \sim y \Leftrightarrow$ there exists an element $g \in G$ such that $y = gx$. Let $x \in X$. Then the equivalence class G_x of x under G or the orbit of x under G is defined by

$$G_x = \{ y \in X : y \sim x \}.$$

How many orbits of x under G are there?

<u>Answers</u>: If G acts transitively on X, then for all $x, y \in X$, there exists an element $g \in G$ such that $y = gx$.

$\therefore$ for all $x \in X$,

$$G_x = \{ y \in X : y = gx \text{ for some } g \text{ in } G \}$$

$\therefore$ $y \sim x$. $\therefore$ $G_y = G_x \Rightarrow$ there is only one orbit of x under G.