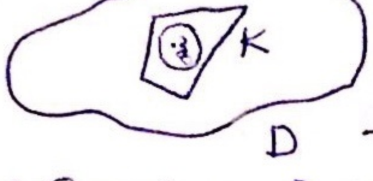
Lecture 13

Another Version of Maximum Modulus Principle

Let f be a holomorphic function on a domain D. Let K be a nonempty compact subset of D. Then $\max_{z\in K} |f(z)|$ is attained at some point in ∂K (the boundary of K).



Proof Write $K = \partial K \cup K^\circ$, where K° is the interior of K. If $K^\circ = \emptyset$, then the proof is finished. Now, suppose that $K^\circ \neq \emptyset$ and $\max_{z\in K} |f(z)| = |f(z_0)|$ for some $z_0 \in K^\circ$. Then there is a neighborhood of z_0 on which

$$|f(z_0)| \geq |f(z)|$$

for all z in the neighborhood. $\therefore$ $|f(z)|$ attains a local maximum in D and this is a contradiction.

Consider a mapping of the form

$$\mathbb{C} \ni z \mapsto cz \in \mathbb{C},$$

where $c \in \mathbb{C}$ with $|c| = 1$. It is called a rotation. Let $c = e^{i\theta}$, where $\theta \in \mathbb{R}$. Then the rotation given above is also of the form

$$\mathbb{C} \ni z \mapsto e^{i\theta} z \in \mathbb{C}.$$

It is a rotation by θ.

Schwarz's Lemma: Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then

1. $|f(z)| \leq |z|$, $z \in \mathbb{D}$,
2. If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D} - \{0\}$, then f is a rotation,
3. $|f'(0)| \leq 1$. If equality holds, then f is a rotation.

Proof: Since $f : \mathbb{D} \to \mathbb{D}$ is holomorphic,

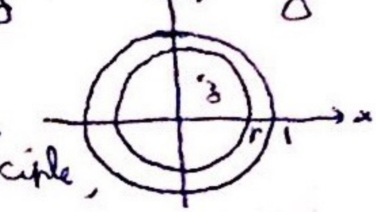
$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \quad z \in \mathbb{D}.$$

Since $f(0) = 0$, $\therefore$ $a_0 = 0$ and

$$f(z) = a_1 z + a_2 z^2 + \cdots, \quad z \in \mathbb{D}.$$

$$\therefore \frac{f(z)}{z} = a_1 + a_2 z + \cdots, \quad z \in \mathbb{D}.$$

$\therefore$ 0 is a removable singularity of $\frac{f(z)}{z}$ and $\frac{f(z)}{z}$ is holomorphic on $\mathbb{D}$. Let $z \in \mathbb{D} - \{0\}$.



Let $r \in (|z|, 1)$. Then by the Maximum Modulus Principle,

$$\left| \frac{f(z)}{z} \right| = \frac{|f(z)|}{|z|} < \frac{1}{r}.$$

Let $r \to 1-$. Then

$$|f(z)| \leq |z|.$$

Next, suppose that there exists a z_0 in $\mathbb{C}$ with

$$|f(z_0)| = |z_0|.$$

Then $\frac{|f(z_0)|}{|z_0|} = 1$. $\therefore$ $\frac{|f(z)|}{|z|}$ attains its local maximum at z_0 in $\mathbb{D}$.

By the Maximum Modulus principle, there exists a constant c such that

$$\frac{f(z)}{z} = c, \quad z \in \mathbb{D}.$$

$\therefore$ $f(z) = cz$, $z \in \mathbb{D}$. But

$$|f(z_0)| = |c||z_0| \Rightarrow |c| = 1.$$

$\therefore$ f is a rotation.

The function g given by
$$g(z) = \frac{f(z)}{z}$$
is holomorphic on $\mathbb{D}$. Note that
$$g(0) = f'(0).$$
$\therefore |f'(0)| \le 1$. If $|f'(0)| = 1$, then $|g(0)| = 1$.
By the Maximum Modulus Principle again, $g(z) = c$, $z \in \mathbb{D}$.
$\therefore |c| = 1 \Rightarrow f(z) = cz \Rightarrow f$ is a rotation.

Automorphism Groups

Let D be a domain in $\mathbb{C}$. A biholomorphism $f : D \to D$ is called an automorphism of D. We denote by $\mathrm{Aut}(D)$ the set of all automorphisms of D. It is a group with respect to the compositions of mappings. We call $\mathrm{Aut}(D)$ the <u>automorphism group</u> of D. We give explicit descriptions of $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$.

$\mathrm{Aut}(\mathbb{D})$

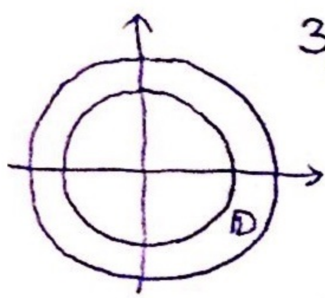
Automorphisms of $\mathbb{D}$

① Let $\theta \in \mathbb{R}$. We define $r_\theta : \mathbb{C} \to \mathbb{C}$ by
$$r_\theta z = e^{i\theta} z, \quad z \in \mathbb{C}.$$
r_θ is a rotation with inverse $r_{-\theta}$.

② Let $\alpha \in \mathbb{C}$ be such that $|\alpha| < 1$. Let $\varphi_\alpha : \mathbb{D} \to \mathbb{C}$ be defined by
$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}, \quad z \in \mathbb{D}.$$
We call φ_α a <u>Möbius transformation</u>.

Observations:

1. ψ_α is holomorphic on $\mathbb{D}$. Obvious. Why?
2. ψ_α is holomorphic on a neighborhood of $\overline{\mathbb{D}}$. To see this, it is obvious for $\alpha = 0$. For $\alpha \neq 0$, let $z \in \mathbb{C}$ be such that
$$|z| \in \left(1, \frac{1}{|\alpha|}\right).$$
Then $1 - \bar{\alpha} z \neq 0$ because $|\bar{\alpha} z| < 1$.



3. ψ_α maps $\mathbb{D}$ into $\mathbb{D}$. To see this, for $|z| = 1$, write $z = e^{i\theta}$.

Then
$$\psi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{1 - \bar{\alpha} e^{i\theta}} = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \bar{\alpha})} = -e^{-i\theta} \frac{\omega}{\bar{\omega}},$$

where
$$\omega = \frac{\alpha - e^{i\theta}}{1} = \alpha - e^{i\theta}.$$

$\therefore |\psi_\alpha(e^{i\theta})| = 1$. $\therefore |\psi_\alpha(z)| = 1, |z| = 1$.

$\therefore$ by the Maximum Modulus Principle,
$$|\psi_\alpha(z)| < 1, \quad z \in \mathbb{D}.$$