Solutions to Assignment 5

22.4. Let $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ be matrices in $\mathrm{SL}(2,\mathbb{R})$.Then for all $z \in \mathbb{H}$,

$$\begin{aligned}
(f_A \circ f_B)(z) &= f_A(f_B z) \\
&= \frac{a_1 B(z) + b_1}{c_1 B(z) + d_1} \\
&= \frac{a_1 \left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1 \left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} \\
&= \frac{(a_1 a_2 + b_1 c_2) z + a_1 b_2 + b_1 d_2}{(c_1 a_2 + c_2 d_1) z + c_1 b_2 + d_1 d_2} \\
&= f_{\begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + c_2 d_1 & c_1 b_2 + d_1 d_2 \end{pmatrix}}(z).
\end{aligned}$$

But

$$AB = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

Therefore

$$f_A \circ f_B = f_{AB}.$$

22.6. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ be such that $f_A(i) = i$. Then

$$\frac{ai + b}{ci + d} = i$$

Therefore

$$ai + b = -c + di.$$

So,

$$b = -c$$

and

$$a = d.$$

Since $ad - bc = 1$, it follows that $a^2 + c^2 = 1$. So, we can let $a = \cos\theta$ and $c = \sin\theta$ for some $\theta \in \mathbb{R}$. Therefore

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Therefore the required isotropy subgroup is

$$\left\{ f_{A_\theta} : A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

22.7. We are given that

$$\mathbb{H} \ni (b, a) \mapsto \begin{pmatrix} \frac{1}{\sqrt{a}} & \frac{b}{\sqrt{a}} \\ 0 & \sqrt{a} \end{pmatrix} \in AN.$$

So, for all (b_1, a_1) and $(b_2, a_2) \in \mathbb{H}$, we have

$$\begin{aligned} \begin{pmatrix} \frac{1}{\sqrt{a_1}} & \frac{b_1}{\sqrt{a_1}} \\ 0 & \sqrt{a_1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{a_2}} & \frac{b_2}{\sqrt{a_2}} \\ 0 & \sqrt{a_2} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{a_1 a_2}} & \frac{b_2}{\sqrt{a_1 a_2}} + \frac{b_1\sqrt{a_2}}{\sqrt{a_1}} \\ 0 & \sqrt{a_1 a_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{a_1 a_2}} & \frac{b_2 + a_2 b_1}{\sqrt{a_1 a_2}} \\ 0 & \sqrt{a_1 a_2} \end{pmatrix}. \end{aligned}$$

So, the group law $\cdot$ is given by

$$(b_1, a_1) \cdot (b_2, a_2) = (b_2 + a_2 b_1, a_1 a_2).$$

23.1. We use the equation (near the top of page 142)

$$u(z) = \frac{1}{\pi i} \int_{\partial\mathbb{D}} \frac{f(w)}{w - z} dw - \overline{a_0}, \quad z \in \mathbb{D},$$

where

$$\operatorname{Re} u(z) = f(z), \quad z \in \partial\mathbb{D}.$$

Therefore for all $z \in \mathbb{C}$ with $|z| = r < 1$,

$$|u(z)| \leq \frac{1}{\pi} \int_0^{2\pi} \frac{M}{1-r} d\theta + |a_0| = |u(0)| + \frac{2M}{1-r}.$$

23.2. We know that all holomorphic functions v on $\mathbb{D}$ with

$$\operatorname{Re} v(re^{i\theta}) \to f(e^{i\theta})$$

uniformly with respect to θ on $[0, 2\pi]$ as $r \to 1-$ are of the form

$$v(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} f(w) \frac{w+z}{w-z} \frac{dw}{w} + ic,$$

wher c is an arbitrary real constant, Since

$$u(re^{i\theta}) = \operatorname{Re} v(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} f(w) \frac{w+z}{w-z} \frac{dw}{w} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} dr,$$

it follows that the function u given by

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\phi$$

is harmionic on $\mathbb{D}$ and is a solution.

23.3. The proof of Theorem 23.8 in the textbook applies.

23.4. Again the proof of Theorem 23.8 can be adapted. Let us look at the details. Let u and v be holomorphic functions on $\mathbb{D}$ such that

$$u(re^{i\theta}) \to f(e^{i\theta})$$

and

$$v(re^{i\theta}) \to f(e^{i\theta})$$

uniformly with respect to θ on $[0, 2\pi]$ as $r \to 1-$. Let $w = u - v$. Then w is a holomorphic function on $\mathbb{D}$ with

$$w(re^{i\theta}) = u(re^{i\theta}) - v(re^{i\theta}) \to f(e^{i\theta}) - f(e^{i\theta}) = 0$$

uniformly with respect to θ on $[0, 2\pi]$ as $r \to 1-$. Suppose that, by way of contradiction, there exists a point z_0 in $\mathbb{D}$ such that $w(z_0) \neq 0$. Then there exists a number $\rho \in (0, 1)$ such that $|z_0| < \rho$ and

$$|w(re^{i\theta})| < \frac{|w(z_0)|}{2}, \quad 0 \leq \theta \leq 2\pi,$$

whenever $\rho \leq r < 1$. Now, by the Maximum Modulus Principle for holomorphic functions, $\max_{|z| \leq \rho} |w(z)|$ has to be attained at some point in the circle $\{z \in \mathbb{C} : |z| = \rho\}$. So, w has a local maximum inside $\mathbb{D}$. By the first version of the Maximum Modulus Principle for holomorphic functions, w must be a constant function on $\mathbb{D}$. But

$$w(re^{i\theta}) \to 0$$

uniformly with respect to θ on $[0, 2\pi]$. Therefore $w = 0$ on $\mathbb{D}$. This proves that $u = v$ on $\mathbb{D}$.

23.5. Let $f(z) = \overline{z}$ for all $z \in \partial\mathbb{D}$. Let

$$u(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a holomorphic function on $\mathbb{D}$ such that

$$u(re^{i\theta}) \to f(e^{i\theta})$$

uniformly with respect to θ on $[0, 2\pi]$ as $r \to 1-$. Therefore

$$u(re^{i\theta}) \to e^{-i\theta}$$

uniformly with respect to θ on $[0, 2\pi]$ as $r \to 1-$. So,

$$\int_0^{2\pi} u(re^{i\theta})\, d\theta \to \int_0^{2\pi} e^{-i\theta} d\theta = 0.$$

as $r \to 1-$. But for all r with $0 < r < 1$,

$$\int_0^{2\pi} u(re^{i\theta})d\theta = \sum_{n=0}^{\infty} a_n r^n \int_0^{2\pi} e^{in\theta} d\theta = a_0.$$

Therefore

$$u(re^{i\theta}) = \sum_{n=1}^{\infty} a_n r^n e^{in\theta} = e^{i\theta} \sum_{n=1}^{\infty} a_n r^n e^{in\theta} = e^{i\theta} \sum_{m=0}^{\infty} a_{m+1} r^{m+1} e^{i(m+1)\theta}.$$

So, let

$$v(re^{i\theta}) = \sum_{m=0}^{\infty} a_{m+1} r^{m+1} e^{i(m+1)\theta}.$$

Then v is a holomorphic function on $\mathbb{D}$ with

$$v(re^{i\theta}) \to e^{-2i\theta}$$

uniformly with respect to θ on $[0, 2\pi]$ as $r \to 1-$. Thus,

$$\int_0^{2\pi} v(re^{i\theta})d\theta \to \int_0^{2\pi} e^{-2i\theta} d\theta = 0.$$

Bur for all $r \in (0, 1)$,

$$\int_0^{2\pi} v(re^{i\theta})\, d\theta = \sum_{m=0}^{\infty} a_{m+1} r^{m+1} \int_0^{\infty} e^{im\theta} d\theta = a_1 r = 0.$$

Therefore $a_1 = 0$. Similarly, $a_n = 0$ for all $n \in \mathbb{N}$. Therefore $u = 0$ on $\mathbb{D}$. So,

$$u(re^{i\theta}) \to 0$$

unformly with respect to θ on $[0, 2\pi]$ as $r \to 1-$. This is a contradiction.