Solutions to Assignment 4

20.1. Let $f(z) = z^2$. Then f is holomorphic on $\mathbb{D}$ and

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2}, \quad n = 2, 3, \ldots$$

Let g be another holomorphic function on $\mathbb{D}$ with

$$g\left(\frac{1}{n}\right) = \frac{1}{n^2}, \quad n = 2, 3, \ldots.$$

Then $f - g$ is holomorphic on $\mathbb{D}$ with $f - g = 0$ on $Z = \{\frac{1}{n} : n = 2, 3, \ldots\}$. But Z has a limit point 0 in $\mathbb{D}$. Therefore $g(z) = f(z)$ for all $z \in \mathbb{D}$. So, the function f is unique.

20.2. Let g be a holomorphic function on $\mathbb{D}$ such that

$$g\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}, \quad n = 2, 3, \ldots.$$

Then using the function f in 20.1,

$$(f - g)\left(\frac{1}{n}\right) = \frac{1}{n^2} - \frac{(-1)^n}{n^2} = \begin{cases} 0, & n = 2, 4, \ldots, \\ \frac{2}{n^2}, & n = 3, 5, \ldots. \end{cases}$$

Therefore $f - g = 0$ on $Z = \{\frac{1}{n} : n = 2, 4, \ldots\}$ that has a limit point 0 in $\mathbb{D}$. Therefore by the Unique Continuation Property of holomorphic functions,

$$f(z) = g(z), \quad z \in \mathbb{D}.$$

But

$$(f - g)\left(\frac{1}{n}\right) = \frac{2}{n^2}, \quad n = 3, 5, \ldots.$$

This is a contradiction.

20.5. Let $g = f \circ F^{-1}$, where $F^{-1} : \mathbb{D} \to \mathbb{H}$ is given by

$$F^{-1}(w) = i\frac{1-w}{1+w}, \quad w \in \mathbb{D}.$$

If $|f(z_0| = 1$ for some $z_0 \in \mathbb{H}$, then $|f(z)|$ attains a local maximum inside $\mathbb{H}$. By the Maximum Modulus Principle for holomorphic functions, $|f(z)|$ is a constant function on $\mathbb{H}$. Since $f(i) = 0$, it follows that $f(z) = 0$ for all $z \in \mathbb{H}$. and the inequality is trivially true. So, assume that $|f(z)| < 1$ for all $z \in \mathbb{H}$. Then

$$g(w) = (f \circ F^{-1})(w) = f\left(i\frac{1-w}{1+w}\right), \quad w \in \mathbb{D}.$$

Therefore $g : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with

$$g(0) = f(i) = 0.$$

By the Schwarz lemma,

$$|g(w)| \leq |w|, \quad w \in \mathbb{D}.$$

So, for all $z \in \mathbb{H}$,

$$|f(z)| = |f(F^{-1}(F(z))| \leq |F(z)| = \left|\frac{i-z}{i+z}\right|.$$

21.2. By Corollary 21.2,

$$K = \{f \in \mathrm{Aut}(\mathbb{D}) : f(0) = 0\} = \{r_\theta : \theta \in \mathbb{R}\}.$$

Next,

$$K \backslash \mathrm{Aut}(\mathbb{D}) = \{Kg : g \in \mathrm{Aut}(\mathbb{D})\}.$$

Let $g = e^{i\theta}\psi_\alpha$ with $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. Then

$$Kg = \{e^{i\phi}e^{i\theta}\psi_\alpha : \phi \in \mathbb{R}\} = \{e^{i\phi}\psi_\alpha : \phi \in \mathbb{R}\} = \mathrm{Aut}(\mathbb{D}).$$

Therefore

$$\{Kg : g \in \mathrm{Aut}(\mathbb{D})\} = \{\mathrm{Aut}(\mathbb{D})\},$$

which can be identified with $\mathbb{D}$.

21.3.(a) The mapping

$$\mathrm{SO}(2,\mathbb{C}) \ni \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \mapsto a \in \mathbb{S}^1$$

is a bijection. Moreover, let $\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}$ and $\begin{pmatrix} b & 0 \\ 0 & \overline{b} \end{pmatrix}$ be elements in $\mathrm{SO}(2,\mathbb{C})$. Then

$$\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & \overline{b} \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & \overline{ab} \end{pmatrix} \mapsto ab \in \mathbb{S}^1.$$

Therefore $\mathrm{SO}(2,\mathbb{C})$ and K are isomorphic groups.

21.3.(b) Let $A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathrm{SU}(1,1)$ be such that

$$f_A(0) = -\frac{a0 + b}{\overline{b}0 + \overline{a}} = 0.$$

Then $b = 0$ and

$$A = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \in \mathrm{SO}(2,\mathbb{C}).$$

Therefore $\mathrm{SO}(2,\mathbb{C})$ is the isotropy subgroup of $\mathrm{SU}(1,1)$.

21.3.(c) By 21.3(a), we see that $\mathrm{SO}(2,\mathbb{C})$ and K are isomorphic groups. Moreover, $\mathrm{Aut}(\mathbb{D})$ and $Z\backslash\mathrm{SU}(1,1)$ are isomorphic groups. In addition,

$$\begin{aligned} Z\backslash\mathrm{SU}(1,1) &= \{Zg : g \in \mathrm{SU}(1,1)\} \\ &= \left\{\pm\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1\right\} = \mathrm{SU}(1,1). \end{aligned}$$

Therefore $\mathrm{SO}(2,\mathbb{C})\backslash\mathrm{SU}(1,1)$ is isomorphic with $K\backslash\mathrm{Aut}(\mathbb{D})$ and hence can be identified with $\mathbb{D}$.

21.4. For all $w \in \mathbb{D}$, let $g = \psi_{f(w)} \circ f \circ \psi_w^{-1}$. Then $g : \mathbb{D} \to \mathbb{D}$ is a holomorphic function such that

$$g(0) = (\psi_{f(w)} \circ f \circ \psi_w^{-1})(0) = \psi_{f(w)}(f(w)) = 0.$$

By the Schwarz lemma, we get for all $z \in \mathbb{D}$,

$$|g(z)| \leq |z|, \quad z \in \mathbb{D}.$$

Therefore

$$|(\psi_{f(w)} \circ f \circ \psi_w^{-1})(\psi_w(z))| = |g(\psi_w(z))| \leq |\psi_w(z)|, \quad z \in \mathbb{D}.$$

So,

$$|\psi_{f(w)}(f(z))| \leq \left| \frac{w - z}{1 - \overline{w}z} \right|, \quad z \in \mathbb{D}.$$

Hence

$$\left| \frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)} \right| \leq \left| \frac{w - z}{1 - \overline{w}z} \right|$$

for all w and z in $\mathbb{D}$. If $f \in \mathrm{Aut}(\mathbb{D})$, then so is f^{-1} and we have

$$\left| \frac{f^{-1}(w) - f^{-1}(z)}{1 - \overline{f^{-1}(w)}f^{-1}(z)} \right| \leq \left| \frac{w - z}{1 - \overline{w}z} \right|$$

for all w and z in $\mathbb{D}$. Replacing w by $f(w)$ and z by $f(z)$ in the preceding inequality, we get

$$\left| \frac{w - z}{1 - \overline{w}z} \right| \leq \left| \frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)} \right|$$

and so we get

$$\left| \frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)} \right| = \left| \frac{w - z}{1 - \overline{w}z} \right|$$

for all w and z in $\mathbb{D}$.

21.5. For all w and z in $\mathbb{D}$, we get by 21.4

$$\left|\frac{f(w)-f(z)}{w-z}\right| \leq \left|\frac{1-\overline{f(w}f(z)}{1-\overline{w}z}\right|.$$

Let $w \to z$. Then

$$|f'(z)| \leq \left|\frac{1-|f(z)|^2}{1-|z|^2}\right|, \quad z \in \mathbb{D},$$

which is

$$\frac{|f'(z)|^2}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.$$