Solutions to Assigment 3

17.1. Let C_ρ be the upper semicircle with center at 0, radius ρ and oriented once in the counterclockwise direction. Let C_r be the upper semicircle with center at 0, radius r and oriented once in the clockwise direction. Here, $r < \rho$. Let $\Gamma_{\rho,r}$ be the contour given by

$$\Gamma_{\rho,r} = C_\rho + [-\rho, -r] - C_r + [r, \rho].$$

Let

$$f(z) = \frac{1 - e^{iz}}{z^2}.$$

Since

$$f(z) = \frac{1}{z^2}\left(1 - 1 - iz - \frac{1}{2!}(iz)^2 - \frac{1}{3!}(iz)^3 - \cdots\right) = \left(-\frac{i}{z} - \frac{1}{2!}i^2 - \frac{1}{31}i^3 z - \cdots\right).$$

Therefore f has a simple pole with residue $-i$ at 0. But 0 is outside $\Gamma_{\rho,r}$. By Cauchy's integral theorem,

$$0 = \int_{\Gamma_{\rho,r}} \frac{1 - e^{iz}}{z^2} dz = \int_{C_\rho} \frac{1 - e^{iz}}{z^2} dz + \int_{-\rho}^{-r} \frac{1 - e^{ix}}{x^2} dx + \int_{C_r} \frac{1 - e^{iz}}{z^2} dz + \int_r^\rho \frac{1 - e^{ix}}{x^2} dx.$$

On C_ρ,

$$\left|\int_{C_\rho} \frac{1 - e^{iz}}{z^2} dz\right| \le \left|\int_{C_\rho} \frac{1}{z^2} dz\right| + \left|\int_{C_\rho} \frac{e^{iz}}{z^2} dz\right|.$$

The first absolute value goes to 0 as $\rho \to \infty$ because the degree of the denominatot is bigger than the degree of the numerator by 2. The second absolute value goes to 0 as $\rho \to \infty$ because of Jordan's lemma, Next,

$$\lim_{r \to 0} \int_{C_r} \frac{1 - e^{iz}}{z^2} dz = -i(\pi - 0)\mathrm{Res}(f, 0) = -\pi.$$

Let $\rho \to \infty$ and $r \to 0$. Then

$$\mathrm{pv} \int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} dx = \pi.$$

Therefore
$$\text{pv} \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} dx = \pi.$$

17.2. Let $\xi = 0$. Let $f(x) = \frac{1}{x}$. Then
$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \left(\text{pv} \int_{-\infty}^{\infty} \frac{1}{x} dx \right) = \left(\lim_{\rho \to \infty} \int_{-\rho}^{\rho} \frac{1}{x} dx \right) = 0$$
because $\frac{1}{x}$ is an odd function. let $\xi < 0$. Then
$$\begin{aligned} \hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \left(\text{pv} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{1}{x} dx \right) \\ &= \lim_{\rho \to \infty, r \to 0} \left(\int_{-\rho}^{-r} e^{-ix\xi} \frac{1}{x} dx + \int_{r}^{\rho} e^{-ix\xi} \frac{1}{x} dx \right). \end{aligned}$$
Let $\Gamma_{\rho,\varepsilon}$ be the contour given by
$$\Gamma_{\rho,r} = C_\rho + [-\rho, r] + C_r + [r, \rho],$$
where C_ρ is the upper semicircle with center at 0, radius ρ and oriented once in the counterclockwise direction; C_r is the upper semicircle with center at 0, radius r and oriented once in the clockwise direction. Let $f(z) = e^{-iz\xi} \frac{1}{z}$. Since f is holomorphic on and inside $\Gamma_{\rho,r}$, by Cauchy's integral theorem,
$$\int_{\Gamma_{\rho,r}} e^{-iz\xi} \frac{1}{z} dz = 0.$$
By Jordan's lemma,
$$\lim_{\rho \to \infty} \int_{C_\rho} e^{-iz\xi} \frac{1}{z} dz = 0.$$
Since 0 is a simple pole of $e^{-iz\xi} \frac{1}{z}$, we have
$$\lim_{r \to 0} \int_{C_r} e^{iz\xi} \frac{1}{z} dz = -i\pi \text{Res}(f, 0) = -i\pi \lim_{z \to 0} (z f(z)) = -\pi i \lim_{z \to 0} e^{-iz\xi} = -\pi i.$$

Therefore

$$0 = \text{pv} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{1}{x} dx - \pi i.$$

So,

$$\text{pv} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{1}{x} dx = \pi i.$$

Hence

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \pi i = \sqrt{\frac{2}{\pi}} i.$$

Let $\xi > 0$. Then we compute $\hat{f}(0)$ by a change of variables. Let $\eta = -\xi$. Then

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \text{pv} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{1}{x} dx = \frac{1}{\sqrt{2\pi}} \text{pv} \int_{-\infty}^{\infty} e^{ix\eta} \frac{1}{x} dx.$$

Let $y = -x$. Then

$$\hat{f}(\xi) = -\frac{1}{\sqrt{2\pi}} \text{pv} \int_{-\infty}^{\infty} e^{-iy\eta} \frac{1}{y} dy = -\sqrt{\frac{2}{\pi}} i.$$

So, for all $\xi \in (-\infty, \infty)$,

$$\hat{f}(\xi) = \sqrt{\frac{2}{\pi}} i \operatorname{sgn}(\xi),$$

where

$$\operatorname{sgn}(\xi) = \begin{cases} 1, & \xi > 0, \\ 0, & \xi = 0, \\ -1, & \xi < 0. \end{cases}$$

18.1. Let

$$f(z) = \frac{\sqrt{z}}{z^2 + 1} = \frac{e^{\frac{1}{2} \log_0 z}}{z^2 + 1} = \frac{e^{\frac{1}{2}(\ln|z| + i \arg_0 z)}}{z^2 + 1}.$$

Let C_ρ be the circle with center 0, radius ρ and oriented once in the counterclockwise direction. Let C_ε be the circle with center 0, radius ε and oriented once in the clocckwise direction. Let $\Gamma_{\rho,\varepsilon}$ be the same contour as the one in Figure 18.1 on page 110 of the textbbook. The function f has two simple poles i and $-i$. Therefore

$$\int_{\Gamma_{\rho,\varepsilon}} f(z)\,dz = 2\pi i(\mathrm{Res}(f,i) + \mathrm{Res}(f,-i)).$$

But

$$\mathrm{Res}(f,i) = \lim_{z\to i}((z-i)f(z)) = \lim_{z\to i}\frac{e^{\frac{1}{2}(\ln|z|+i\arg_0 z)}}{z+i} = \frac{e^{i\frac{\pi}{4}}}{2i} = \frac{1}{2\sqrt{2}} - i\frac{1}{2\sqrt{2}}$$

and

$$\mathrm{Res}(f,-i) = \lim_{z\to -i}((z+i)f(z)) = \lim_{z\to -i}\frac{e^{\frac{1}{2}(\ln|z|+i\arg_0 z)}}{z-i} = \frac{e^{i\frac{3\pi}{4}}}{-2i} = -\frac{1}{2\sqrt{2}} - i\frac{1}{2\sqrt{2}}.$$

Therefore

$$\int_{\Gamma_{\rho,\varepsilon}} f(z)\,dz = 2\pi i\left(-i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\pi.$$

Now,

$$\int_{\gamma_+} \frac{\sqrt{z}}{z^2+1}dz = \int_\varepsilon^\rho \frac{\sqrt{x}}{x^2+1}dx$$

and

$$\int_{\gamma_-} \frac{\sqrt{z}}{z^2+1}dz = -\int_\varepsilon^\rho \frac{-\sqrt{x}}{x^2+1}dx = \int_\varepsilon^\rho \frac{\sqrt{x}}{x^2+1}dx.$$

Next,

$$\left|\int_{C_\rho} \frac{\sqrt{z}}{z^2+1}dz\right| \le 2\pi\rho\frac{\rho^{1/2}}{\rho^2-1} \to 0$$

as $\rho\to\infty$ and

$$\left|\int_{C_\varepsilon} \frac{\sqrt{z}}{z^2+1}dz\right| \le 2\pi\varepsilon\frac{\varepsilon^{1/2}}{1-\varepsilon^2} \to 0$$

as $\varepsilon\to 0$. So, letting $\rho\to\infty$ and $\varepsilon\to 0$, we get

$$\sqrt{2}\pi = 2\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx.$$

Therefore
$$\int_{-\infty}^{\infty} \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi}{\sqrt{2}}.$$

18.2. Let
$$f(z) = \frac{z^\alpha}{(z+9)^2} = \frac{e^{\alpha \log_0 z}}{(z+9)^2} = \frac{e^{\alpha (\ln|z| + i \arg_0 z)}}{(z+9)^2}.$$
Let $\Gamma_{\rho,\varepsilon}$ be the contour considered in Figure 18.1 on page 110. Then f has a pole of order 2 at -9 and

$$\mathrm{Res}(f,-9) = \lim_{z\to -9} \frac{d}{dz}((z+9)^2 f(z)) = \lim_{z\to -9} \frac{d}{dz}(z^\alpha) = \lim_{z\to -9} \alpha z^{\alpha-1} = \alpha 9^{\alpha-1} e^{i(\alpha-1)\pi}.$$

By Cauchy's residue theorem,
$$\int_{\Gamma_{\rho,\varepsilon}} \frac{z^\alpha}{(z+9)^2}dz = 2\pi i \mathrm{Res}(f,-9) = 2\pi i \alpha 9^{\alpha-1} e^{i(\alpha-1)\pi}.$$

Now,
$$\int_{\gamma_+} \frac{z^\alpha}{(z+9)^2}dz = \int_\varepsilon^\rho \frac{x^\alpha}{(x+9)^2}dx$$
and
$$\int_{\gamma_-} \frac{z^\alpha}{(z+9)^2}dz = -\int_\varepsilon^\rho \frac{x^\alpha e^{2\pi i \alpha}}{(x+9)^2}dx.$$
Next,
$$\left| \int_{C_\rho} \frac{z^\alpha}{(z+9)^2}dz \right| \leq 2\pi\rho \frac{\rho^\alpha}{(\rho-9)^2} \to 0$$
as $\rho \to \infty$. Also,
$$\left| \int_{C_\varepsilon} \frac{z^\alpha}{(z+9)^2}dz \right| \leq 2\pi\varepsilon \frac{\varepsilon^\alpha}{(9-\varepsilon)^2} \to 0$$
as $\varepsilon \to 0$. So, letting $\rho \to \infty$ and $\varepsilon \to 0$, we get
$$2\pi i \alpha 9^{\alpha-1} e^{i(\alpha-1)\pi} = (1 - e^{2\pi i \alpha}) \int_0^\infty \frac{x^\alpha}{(x+9)^2}dx.$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{x^\alpha}{(x+9)^2}dx &= \frac{2\pi i\alpha 9^{\alpha-1}e^{i(\alpha-1)\pi}}{1-e^{2\pi i\alpha}} \\ &= \frac{-2\pi i\alpha 9^{\alpha-1}e^{i\alpha\pi}}{e^{i\pi\alpha}(e^{-i\pi\alpha}-e^{i\pi\alpha})} \\ &= \frac{9^{\alpha-1}\pi\alpha}{\sin(\pi\alpha)}.\end{aligned}$$

18.3. Let

$$f(z) = e^{-z}z^{\alpha-1} = e^{-z}e^{(\alpha-1)\log_{-\pi} z} = e^{-z}e^{(\alpha-1)(\ln|z|+i\arg_{-\pi} z)}.$$

Let $\Gamma_{\rho,\varepsilon}$ be the contour in Figure 18.3 on page 113 of the textbook. Then by Cauchy's integral theorem,

$$\int_{\Gamma_{\rho,\varepsilon}} e^{-z}z^{\alpha-1}dz = 0.$$

We have

$$\int_{\rho,\varepsilon,2} e^{-z}z^{\alpha-1}dz = \int_\varepsilon^\rho e^{-x}x^{\alpha-1}dx.$$

Now, parametrizing $\gamma_{\rho,\varepsilon,1}$ by

$$z = iy,$$

where y goes down from ρ to ε, we have

$$\int_{\gamma_{\rho,\varepsilon,1}} e^{-z}z^{\alpha-1}dz = -\int_\varepsilon^\rho e^{-iy}e^{(\alpha-1)(\ln y+i\pi/2)}idy = -\int_\varepsilon^\rho e^{-iy}y^{\alpha-1}e^{i(\alpha-1)\pi/2}idy.$$

Next, parametrizing C_ρ by

$$z = \rho e^{i\theta}, \quad 0 \le \theta \le \frac{\pi}{2},$$

we get

$$\left|\int_{C_\rho} e^{-z}z^{\alpha-1}dz\right| \le \frac{\pi}{2}\rho e^{-\rho\cos\theta}\rho^{\alpha-1} = \frac{\pi}{2}e^{-\rho\cos\theta}\rho.$$

By looking at the graphs of $y = \cos\theta$ and $y = -\frac{2}{\pi}\left(\theta - \frac{\pi}{2}\right)$ on $[0, \pi/2)$, we see that

$$e^{-\rho\cos\theta} \le e^{-\rho\left(1-\frac{2}{\pi}\theta\right)}, \quad 0 \le \theta < \pi/2.$$

Therefore

$$\left|\int_{C_\rho} e^{-z} z^{\alpha-1} dz\right| \le \frac{\pi}{2}\rho e^{-\rho\left(1-\frac{2}{\pi}\theta\right)} \to 0$$

as $\rho \to \infty$. Similarly,

$$\left|\int_{C_\varepsilon} e^{-z} z^{\alpha-1} dz\right| \le \frac{\pi}{2}\varepsilon e^{-\varepsilon\cos\theta}\varepsilon^{\alpha-1} \to 0$$

as $\varepsilon \to 0$. Letting $\rho \to \infty$ and $\varepsilon \to 0$, we get

$$\int_0^\infty e^{-ix} x^{\alpha-1} dx = \Gamma(\alpha) e^{-i\alpha\pi/2}.$$

Taking the real part on each side, we get

$$\int_0^\infty x^{\alpha-1}\cos x\, dx = \cos\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha).$$

19.1. For all $z \in \{z \in \mathbb{C} : |z - i| < 1\}$, let $T_i(z) = z - i$. Let $f : \{z \in \mathbb{C} : |z - i| < 1\} \to \mathbb{C}$ be defined by

$$f = r_{-\frac{\pi}{2}} \circ F^{-1} \circ T_i,$$

where F is the Cayley transform. Then f is a biholomorphism from $\{z \in \mathbb{C} : |z - i| < 1\}$ onto $\{w \in \mathbb{C} : \operatorname{Re} w > 0\}$.

19.2. The function $f : \mathbb{H} \to \{w \in \mathbb{C} : 0 < \arg w < \frac{\pi}{2}\}$ given by

$$f(z) = z^{1/2} = e^{\frac{1}{2}\operatorname{Log} z}, \quad z \in \mathbb{H},$$

is a biholomorphism with inverse g given by

$$g(w) = w^2$$

for all $w \in \{w \in \mathbb{C} : 0 < \arg w < \frac{\pi}{2}\}$.