Solutions to Assigment 2

14.1. Let C be the unit circle centered at the origin and oriented once in the counterclockwise direction. Parametrizing C by

$$z = e^{i\theta}, \quad 0 \le \theta \le 2\pi,$$

we get

$$\begin{aligned} & \int_0^{2\pi} \frac{d\theta}{2 + \sin\theta} \\ = & \int_C \frac{1}{2 + \frac{1}{2i}\left(z - \frac{1}{z}\right)} \frac{1}{iz} dz \\ = & \int_C \frac{2}{z^2 + 4iz - 1} dz. \\ = & \int_C \frac{2}{(z - z_1)(z - z_2)} dz, \end{aligned}$$

where $z_1 = (-2 + \sqrt{3})i$ and $z_2 = (-2 - \sqrt{3})i$. Let

$$f(z) = \frac{2}{z^2 + 4iz - 1} = \frac{2}{(z - z_1)(z - z_2)}.$$

Then z_1 and z_2 are simple poles of f. Since only z_1 is inside C, it follows from Cauchy's residue theorem that

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{2 + \sin\theta} &= \int_C \frac{2}{(z - z_1)(z - z_2)} dz \\ &= 2\pi i \text{Res}(f, z_1) \\ &= 2\pi i \lim_{z \to z_1} (z - z_1) f(z) \\ &= 2\pi i \lim_{z \to z_1} \frac{2}{z - z_2} \\ &= 2\pi i \frac{2}{z_1 - z_2} = 2\pi i \frac{2}{2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}. \end{aligned}$$

14.2. Since $\frac{1}{(3+2\cos\theta)^2}$ is an even function of θ,

$$\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(3+2\cos\theta)^2}.$$

Let C be the unit circle centered at the origin and oriented once in the counterclockwise direction. Then

$$\begin{aligned}
\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} \\
&= \frac{1}{2}\int_C \frac{1}{\left(3+\left(z+\frac{1}{z}\right)\right)^2}\frac{1}{iz}dz \\
&= \frac{1}{2i}\int_C \frac{z}{(z^2+3z+1)^2}dz \\
&= \frac{1}{2i}\int_C \frac{z}{(z-z_1)^2(z-z_2)^2}dz,
\end{aligned}$$

where $z_1 = \frac{-3+\sqrt{5}}{2}$ and $z_2 = \frac{-3-\sqrt{5}}{2}$ are poles of order 2 of the function

$$f(z) = \frac{z}{(z-z_1)^2(z-z_2)^2}.$$

Since only the pole z_1 is inside C, we can use Cauchy's residue theorem to get

$$\begin{aligned}
\int_C \frac{z}{(z-z_1)^2(z-z_2)^2}dz &= 2\pi i \mathrm{Res}(f, z_1) \\
&= 2\pi i \lim_{z\to z_1} \frac{d}{dz}((z-z_1)^2 f(z)) \\
&= 2\pi i \lim_{z\to z_1} \frac{d}{dz}\left(\frac{z}{(z-z_2)^2}\right) \\
&= 2\pi i \lim_{z\to z_1} \frac{(z-z_2)^2 - 2z(z-z_2)}{(z-z_2)^4} \\
&= 2\pi i \lim_{z\to z_1} \frac{-z-z_2}{(z-z_2)^3} = 2\pi i \frac{3}{5\sqrt{5}}.
\end{aligned}$$

Therefore
$$\int_0^{\pi} \frac{d\theta}{(3+2\cos\,\theta)^2} = \frac{1}{2i} 2\pi i \frac{3}{5\sqrt{5}} = \frac{3\pi}{5\sqrt{5}}.$$

14.4. Let $b = \frac{1}{a}$, where $|a| > 1$. By 14.3,
$$\int_0^{2\pi} \frac{1}{1-2b\cos\,\theta + b^2} d\,\theta = \frac{2\pi}{1-b^2}.$$
Therefore
$$\int_0^{2\pi} \frac{1}{1-2\left(\frac{1}{a}\right)\cos\,\theta + \left(\frac{1}{a}\right)^2} d\,\theta = \frac{2\pi}{1-\frac{1}{a^2}}.$$
So,
$$\int_0^{2\pi} \frac{a^2}{1-2a\cos\,\theta + a^2} d\,\theta = \frac{2\pi a^2}{a^2-1},$$
which is the same as
$$\int_0^{2\pi} \frac{1}{1-2a\cos\,\theta + a^2} d\,\theta = \frac{2\pi}{a^2-1}.$$

14.5. We first assume that $a \neq 0$. Let C be the unit circle centered at the origin and oriented once in the counterclockwise direction. Parametrizing C by
$$z = e^{i\theta}, \quad 0 \le \theta \le 2\pi,$$
we get
$$\begin{aligned} & \int_0^{2\pi} \frac{\cos\,\theta}{1-2a\cos\,\theta + a^2} d\,\theta \\ = & \int_C \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{1-a\left(z+\frac{1}{z}\right)+a^2} \frac{1}{iz} dz \\ = & -\frac{1}{2ia} \int_C \frac{z^2+1}{z(z-a)\left(z-\frac{1}{a}\right)} dz. \end{aligned}$$

Let
$$f(z) = \frac{z^2+1}{z(z-a)\left(z-\frac{1}{a}\right)}.$$
Then f has three simple poles 0, a and $\frac{1}{a}$. Only 0 and a are inside C. So,
$$\int_C \frac{z^2+1}{z(z-a)\left(z-\frac{1}{a}\right)} dz = 2\pi i(\text{Res}(f,0) + \text{Res}(f,a)).$$
But
$$\text{Res}(f,0) = \lim_{z\to 0}(zf(z)) = \lim_{z\to 0} \frac{z^2+1}{(z-a)\left(z-\frac{1}{a}\right)} = 1$$
and
$$\text{Res}(f,a) = \lim_{z\to a}((z-a)f(z)) = \lim_{z\to a} \frac{z^2+1}{z\left(z-\frac{1}{a}\right)} = \frac{a^2+1}{a\left(a-\frac{1}{a}\right)}.$$
Therefore
$$\int_C \frac{z^2+1}{z(z-a)\left(z-\frac{1}{a}\right)} dz = 2\pi i\left(1 + \frac{a^2+1}{a^2-1}\right) = 2\pi i \frac{2a^2}{a^2-1}.$$
So,
$$\int_0^{2\pi} \frac{\cos\,\theta}{1-2a\cos\,\theta + a^2} d\theta = -\frac{1}{2ia} 2\pi i \frac{2a^2}{a^2-1} = \frac{2\pi a}{1-a^2}.$$
If $a = 0$, then
$$\int_0^{2\pi} \frac{1}{1-2a\cos\,\theta + a^2} d\theta = \int_0^{2\pi} \cos\,\theta\, d\,\theta = 0 = \frac{2\pi a}{1-a^2}.$$

15.1. By definitition,
$$\text{pv}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} \frac{dx}{x^2+2x+2}.$$
Let C_ρ^+ be the upper semicircle with center at the origin, radius ρ and oriented once in the counterclockwise direction. Let Γ_ρ be the contour given by
$$\Gamma_\rho = C_\rho^+ + [-\rho, \rho].$$

Let

$$f(z) = \frac{1}{z^2+2z+2} = \frac{1}{(z-z_+)(z-z_-)},$$

where $z_+ = -1+i$ and $z_- = -1-i$. z_+ and z_- are simple poles of f and only z_+ is inside Γ_+. Therefore by Cauchy's residue theorem,

$$\begin{aligned}\int_{\Gamma_\rho} f(z)\,dz &= 2\pi i \mathrm{Res}(f, z_+) = 2\pi i \lim_{z\to z_+}((z-z_+)f(z)) \\ &= 2\pi i \lim_{z\to z_+} \frac{1}{z-z_-} = 2\pi i \frac{1}{z_+-z_-} = \pi.\end{aligned}$$

Let $\rho \to \infty$. Then by Lemma 15.4,

$$\lim_{\rho\to\infty} \int_{C_\rho^+} f(z)\,dz = 0.$$

So,

$$\mathrm{pv}\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}dx = \lim_{\rho\to-\infty} \int_{-\rho}^{\rho} \frac{1}{x^2+2x+2}dx = \pi.$$

16.1. We begin with

$$\mathrm{pv}\int_{-\infty}^{\infty} \frac{e^{i2x}}{x-3i}dx = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{e^{i2x}}{x-3i}dx.$$

Let Γ_ρ^+ be the same contour as in 15.1. Let

$$f(z) = \frac{e^{iz2}}{z-3i}.$$

Then $3i$ is the only isolated singularity of f and it is a simple pole. So,

$$\int_{\Gamma_\rho^+} \frac{e^{iz2}}{z-3i}dz = 2\pi i \mathrm{Res}(f, 3i) = 2\pi i \lim_{z\to 3i} e^{iz2} = 2\pi i e^{-6}.$$

By Jordan's lemma,

$$\lim_{\rho\to\infty} \int_{C_\rho^+} \frac{e^{iz2}}{z-3i}dz = 0.$$

Therefore

$$\text{pv}\int_{-\infty}^{\infty} \frac{e^{i2x}}{x-3i}dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} \frac{e^{i2x}}{x-3i}dx = 2\pi i e^{-6}.$$

On the other hand, we let C_ρ^- be the lower semicircle with center at the origin, radius ρ and oriented once in the counterclockwise direction. Let Γ_ρ^- be the contour given by

$$C_\rho^- - [-\rho, \rho].$$

Let $g(z) = \frac{e^{-iz2}}{z-3i}$. Then the only isolated singularity of g is $3i$ and it is not inside Γ_ρ^-. So, by Cauchy's integral theorem,

$$\int_{\Gamma_\rho^-} \frac{e^{-iz2}}{z-3i}dz = 0.$$

By Jordan's lemma again,

$$\lim_{\rho\to\infty}\int_{C_\rho^-} \frac{e^{-iz2}}{z-3i}dz = 0.$$

Therefore

$$\text{pv}\int_{-\infty}^{\infty} \frac{e^{-i2x}}{x-3i}dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} \frac{e^{-i2x}}{x-3i}dx = 0.$$

So,

$$\text{pv}\int_{-\infty}^{\infty} \frac{\cos(2x)}{x-3i}dx = \frac{1}{2}\left(\text{pv}\int_{-\infty}^{\infty} \frac{e^{i2x}}{x-3i}dx + \text{pv}\int_{-\infty}^{\infty} \frac{e^{-i2x}}{x-3i}dx\right) = \pi i e^{-6}.$$

16.2. Let

$$f(x) = \frac{1}{x+i}.$$

Then

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\text{pv}\int_{-\infty}^{\infty} e^{-ix0}\frac{1}{x+i}dx = \frac{1}{\sqrt{2\pi}}\text{pv}\int_{-\infty}^{\infty} \frac{1}{x+i}dx.$$

But

$$\text{pv}\int_{-\infty}^{\infty}\frac{1}{x+i}dx = \text{pv}\int_{-\infty}^{\infty}\frac{x-i}{x^2+1}dx = \text{pv}\left(\int_{-\infty}^{\infty}\frac{x}{x^2+1}dx - i\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx\right).$$

Since $\frac{x}{x^2+1}$ is an odd function of x, we have

$$\text{pv}\int_{-\infty}^{\infty}\frac{x}{x^2+1}dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho}\frac{x}{x^2+1}dx = 0.$$

It has been calculated in class that $\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx = \pi$. So,

$$\hat{f}(0) = -\frac{1}{\sqrt{2\pi}}\pi i = -\sqrt{\frac{\pi}{2}}i.$$

Let $\xi < 0$. Then

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\text{pv}\int_{-\infty}^{\infty}e^{-ix\xi}\frac{1}{x+i}dx.$$

Let C_ρ^+ be the upper semicircle with center at the origin, radius ρ and oriented once in the counterclockwise direction. Let Γ_ρ^+ be the contour given by

$$\Gamma_\rho^+ = C_\rho^+ + [-\rho,\rho].$$

Let

$$f(z) = e^{-iz\xi}\frac{1}{z+i}.$$

Then $-i$ is the simple pole of f and it lies outside Γ_ρ^+ So, by Cauchy's integral theorem,

$$\int_{\Gamma_\rho^+}e^{-iz\xi}\frac{1}{z+i}dz = 0.$$

By Jordan's lemma,

$$\lim_{\rho\to\infty}\int_{C_\rho^+}e^{-iz\xi}\frac{1}{z+i}dz = 0.$$

Therefore

$$\mathrm{pv}\int_{-\infty}^{\infty} e^{-ix\xi}\frac{1}{x+i}dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} e^{-ix\xi}\frac{1}{x+i}dx = 0.$$

So,

$$\hat{f}(\xi) = 0, \quad \xi < 0.$$

Finally, let $\xi > 0$. Let C_ρ^- be the lower semicircle withe center at the origin, radius ρ and oriented once in the counterclockwise direction. Let Γ_ρ^- be the contour given by

$$\Gamma_\rho^- = C_\rho^- - [-\rho,\rho].$$

Since $-i$ is the only simple pole of g inside Γ_ρ^-, we have

$$\begin{aligned}
\int_{\Gamma_\rho^-} e^{-iz\xi}\frac{1}{z+i}dx &= 2\pi i \mathrm{Res}(g,-i) \\
&= 2\pi i \lim_{z\to -i}((z+i)g(z)) \\
&= 2\pi i \lim_{z\to -i} e^{-iz\xi} \\
&= 2\pi i e^{-\xi}.
\end{aligned}$$

By Jordan's lemma,

$$\lim_{\rho\to\infty}\int_{C_\rho^-} e^{-iz\xi}\frac{1}{z+i}dz = 0.$$

Therefore

$$\mathrm{pv}\int_{-\infty}^{\infty} e^{-ix\xi}\frac{1}{x+i}dx = 2\pi i e^{-\xi}.$$

So,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}2\pi e^{\xi} = \sqrt{2\pi} i e^{-\xi}.$$

16.3. Let $f(x) = \frac{1}{1+x^2}$. Then

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\mathrm{pv}\int_{-\infty}^{\infty} e^{-ix0}\frac{1}{1+x^2}dx = \frac{1}{\sqrt{2\pi}}\pi = \sqrt{\frac{\pi}{2}}.$$

Let $\xi < 0$. Let C_ρ^+ be the upper semicircle with center at the origin, radius ρ and oriented once in the counterclockwise direction. Let Γ_ρ^+ be the contour given by

$$\Gamma_\rho^+ = C_\rho^+ + [-\rho, \rho].$$

Let

$$f(z) = e^{-iz\xi} \frac{1}{1+z^2}.$$

Then f has two simple poles i and $-i$ and i is inside Γ_ρ^+. Using Cauchy's residue theorem,

$$\begin{aligned} \int_{\Gamma_\rho^+} e^{-iz\xi} \frac{1}{1+z^2} dz &= 2\pi i \operatorname{Res}(f, i) \\ &= 2\pi i \lim_{z \to i} ((z-i) f(z)) \\ &= 2\pi i \lim_{z \to i} e^{-iz\xi} \frac{1}{z+i} \\ &= \pi e^{i\xi} = \pi e^{-|\xi|}. \end{aligned}$$

By Jordan's lemma,

$$\lim_{\rho \to \infty} \int_{C_\rho^+} e^{-iz\xi} \frac{1}{1+z^2} dz = 0.$$

Therefore

$$\text{pv} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{1}{1+x^2} dx = \lim_{\rho \to \infty} \int_{-\rho}^{\rho} e^{-ix\xi} \frac{1}{1+x^2} dx = \pi e^{-|\xi|}.$$

So,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \pi e^{-|\xi|} = \sqrt{\frac{\pi}{2}} e^{-|\xi|}.$$

Now, let $\xi > 0$. For a change, let us use a change of variables to compute $\hat{f}(\xi)$. Let $t = -x$. Then

$$\text{pv} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{1}{1+x^2} dx = \text{pv} \int_{-\infty}^{\infty} e^{-it(-\xi)} \frac{1}{1+t^2} dt = \pi e^{-|-\xi|} = \pi e^{-|\xi|}.$$

Therefore

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{2}} e^{-|\xi|}.$$

Hence for all $\xi \in (-\infty, \infty)$,

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{2}} e^{-|\xi|}.$$