Lecture 6

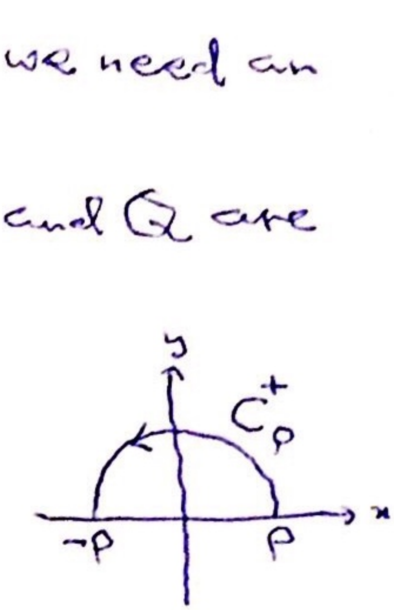
To compute Cauchy principal values, we need an easy lemma on limits.

Lemma: Let $f(z) = \dfrac{P(z)}{Q(z)}$, where P and Q are polynomials with

$$\deg(Q) - \deg(P) \geq 2.$$

Then

$$\lim_{\rho \to \infty} \int_{C_\rho^+} f(z)\,dz = 0,$$



where C_ρ^+ is the upper semicircle with center 0, radius ρ and oriented once in the counterclockwise direction.

Proof: Write

$$f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}, \quad n - m \geq 2.$$

On C_ρ^+,

$$|f(z)| = \frac{|z|^m \left|\left(\frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \cdots + a_m\right)\right|}{|z|^n \left|\left(\frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \cdots + b_n\right)\right|} \leq |z|^{-(n-m)} \left(\frac{|a_m|}{|b_n|} + 1\right)$$

when z is sufficiently large, say, $|z| \geq R$. So,

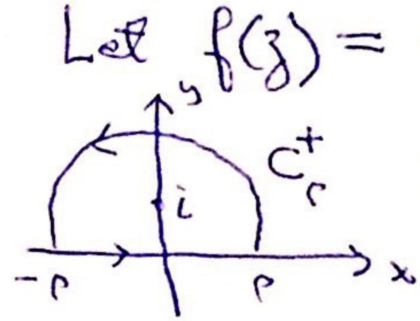
$$\left| \int_{C_\rho^+} f(z)\,dz \right| \leq \pi \rho^{-(n+1-m)} \left(\frac{|a_m|}{|b_n|} + 1\right) \to 0 \text{ as } \rho \to \infty$$

because

$$n + 1 - m \geq 1.$$

Example: Compute $\mathrm{pv}\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ using residues

Solution Let $f(z) = \dfrac{1}{1+z^2}$. Let C_ρ^+ be as before.



Let $\Gamma_\rho = C_\rho^+ + [-\rho, \rho]$.

Then

$$\int_{\Gamma_\rho} f(z)\,dz = \int_{C_\rho^+} f(z)\,dz + \int_{-\rho}^{\rho} f(x)\,dx.$$

But

$$\int_{\Gamma_\rho} f(z)\,dz = \int_{\Gamma_\rho} \frac{1}{(z+i)(z-i)}\,dz.$$

So, f has a simple pole i inside Γ_ρ. $\therefore$

$$\int_{\Gamma_\rho} f(z)\,dz = 2\pi i \operatorname{Res}(f, i)$$

$$= 2\pi i \lim_{z\to i}(z-i) f(z) = 2\pi i \lim_{z\to i} \frac{1}{z+i}$$

$$= \pi.$$

$$\therefore \pi = \int_{C_\rho^+} f(z)\,dz + \int_{-\rho}^{\rho} f(x)\,dx.$$

Letting $\rho \to \infty$.

By the lemma,

$$\pi = \lim_{\rho\to\infty} \int_{C_\rho^+} f(z)\,dz + \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx$$

$$= \mathrm{pv}\int_{-\infty}^{\infty} f(x)\,dx = \mathrm{pv}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

To compute more Cauchy principal values, we need a more powerful lemma than the previous one.

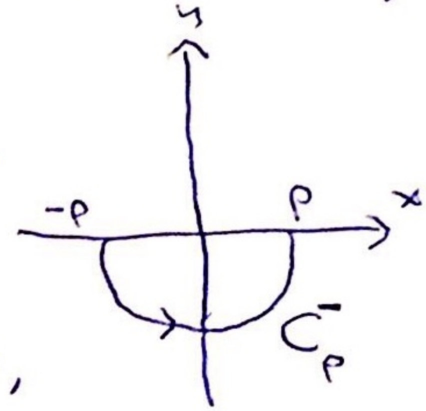
Lemma (Jordan's Lemma) Let $f(z)=\dfrac{P(z)}{Q(z)}$, where P and Q are polynomials with

$$\deg(Q)-\deg(P)\geq 1. \quad (\text{instead of } 2)$$

Then for all $\xi>0$,

$$\lim_{\rho\to\infty}\int_{C_\rho^+} e^{iz\xi} f(z)\,dz=0.$$

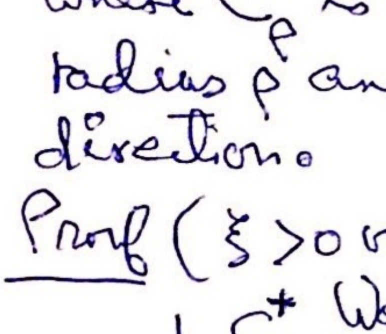
If $\xi<0$, then



$$\lim_{\rho\to\infty}\int_{C_\rho^-} e^{iz\xi} f(z)\,dz=0,$$

where C_ρ^- is the lower semicircle with center at 0, radius ρ and oriented once in the counterclockwise direction.

Proof ($\xi>0$ only) Parametrize C_ρ^+ by $z=\rho e^{it}$, $0\leq t\leq\pi$. We get



$$\int_{C_\rho^+} e^{iz\xi}\frac{P(z)}{Q(z)}\,dz=\int_0^\pi e^{i\xi(\rho e^{it})}\frac{P(\rho e^{it})}{Q(\rho e^{it})}\,i\rho e^{it}\,dt.$$

Let $g(t)=e^{i\xi(\rho e^{it})}\dfrac{P(\rho e^{it})}{Q(\rho e^{it})}\,i\rho e^{it}$, $t\in[0,\pi]$.

Note that on C_ρ^+,

$$\left|e^{i\xi(\rho e^{it})}\right|=\left|e^{i\xi\rho\cos t-\xi\rho\sin t}\right|=e^{-\xi\rho\sin t},$$

$$\left|\frac{P(z)}{Q(z)}\right|\leq\left(\frac{|a_m|}{|b_n|}+1\right)|z|^{n-m}=\left(\frac{|a_m|}{|b_n|}+1\right)\rho^{n-m}.$$

So $|g(t)|\leq e^{-\xi\rho\sin t}\left(\dfrac{|a_m|}{|b_n|}+1\right)\rho^{n-m+1}$

$$\therefore \left| \int_{C_\rho^+} e^{iz\xi} \frac{P(z)}{Q(z)} dz \right| \leq \left(\frac{|a_m|}{|b_n|} + 1 \right) \rho^{n-m+1} \int_0^\pi e^{-\xi\rho \sin t} dt.$$

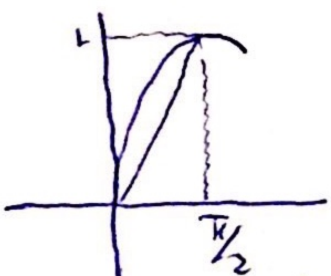
But $\displaystyle\int_0^\pi e^{-\xi\rho\sin t} dt = \int_0^{\pi/2} e^{-\xi\rho\sin t} dt + \int_{\pi/2}^{\pi} e^{-\xi\rho\sin t} dt.$

Let $t = \pi - s$ in the second integral on the right hand side.

Then $\displaystyle\int_{\pi/2}^{\pi} e^{-\xi\rho\sin t} dt = -\int_{\pi/2}^{0} e^{-\xi\rho \sin(\pi-s)} ds = \int_0^{\pi/2} e^{-\xi\rho\sin t} dt.$

So $\displaystyle\int_0^\pi e^{-\xi\rho\sin t} dt = 2\int_0^{\pi/2} e^{-\xi\rho\sin t} dt.$

Since $\sin t \geq \frac{2}{\pi} t$, $t \in [0, \frac{\pi}{2}]$,



we get

$$\int_0^\pi e^{-\xi\rho\sin t} dt \leq 2\int_0^{\frac{\pi}{2}} e^{-(\xi\rho 2/\pi)t} dt$$

$$= 2\left(-\frac{e^{-(\xi\rho 2/\pi)t}}{(\xi\rho 2)/\pi} \right)\Bigg|_0^{\pi/2}$$

$$= \frac{\pi}{\xi\rho}\left(1 - e^{-\xi\rho}\right) \to 0$$

as $\rho \to \infty$.

So $\displaystyle\int_{C_\rho^+} e^{iz\xi} f(z) dz \to 0$ as $\rho \to \infty$.