Lecture 9

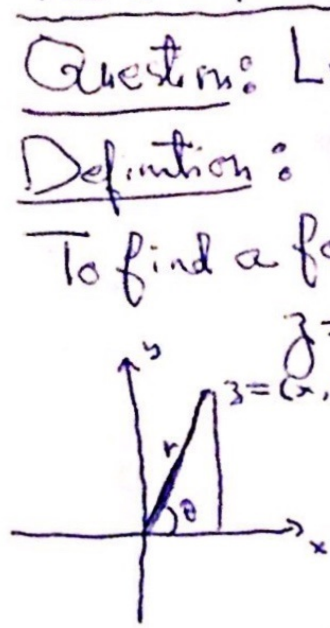
The Complex Logarithm

Question: Let $z \in \mathbb{C} - \{0\}$. What is $\log z$?

Definition: Let $w = \log z$. Then $z = e^{w}$.

To find a formula for $\log z$, let $w = u + iv$ and

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}, \text{ where}$$
$$r = |z|.$$



$$\therefore re^{i\theta} = e^{w} = e^{u}e^{iv}.$$

$\therefore r = e^{u}$, $\theta = v$ $\therefore \log z = \ln|z| + i\theta = \ln|z| + i\arg z$.

$\therefore \log$ is not a function, it is a multi-valued function.

In fact, let θ be a branch of $\arg z$, then

$$\log z = \ln|z| + i\theta + i2k\pi, \ k \in \mathbb{Z}.$$

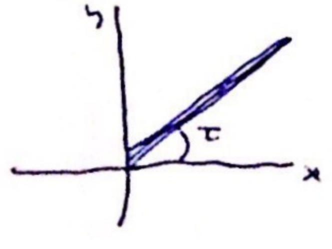
Definition: Let $z \in \mathbb{C} - \{0\}$. Let $\tau \in \mathbb{R}$. Then the branch $\log_\tau z$ of $\log z$ is given by

$$\log_\tau z = \ln|z| + i\arg_\tau z.$$

If $\tau = -\pi$, then $\log_{-\pi}$ is the principal branch of $\log$, denoted by $\operatorname{Log} z$.

Theorem: Let $\mathbb{C}^{o}_{\tau} = \mathbb{C} - \{(r,\tau) : r > 0\}$. Then $\log_\tau z$ is holomorphic on $\mathbb{C}^{o}_{\tau}$ and

$$\frac{d}{dz}\log_\tau z = \frac{1}{z}, \ z \in \mathbb{C}^{o}_{\tau}.$$



Definition: Let $z \in \mathbb{C} - \{0\}$. Let $\alpha \in \mathbb{C}$. Then we define z^α by

$$z^\alpha = e^{\alpha \log z}.$$

This is again a multi-valued function, not a function.

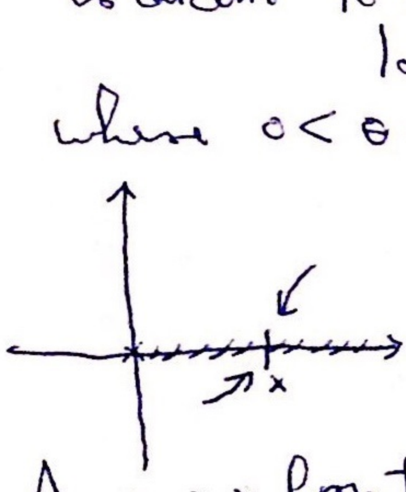
Let $\tau \in \mathbb{R}$. Then the branch $e^{\alpha \operatorname{Log} z}$ is called the principal branch of z^α.

Back to Chapter 18: (Integrals on Branch Cuts).

Let $\alpha \in \mathbb{R} - \mathbb{Z}$, i.e., α is a real number but not an integer. Let $f(z) = z^\alpha = e^{\alpha \log_0 z}$, where the branch $e^{\alpha \log_0 z}$ is taken. To recall,

$$\log_0 z = \ln|z| + i\theta,$$

where $0 < \theta \le 2\pi$. Then f is holomorphic on $\mathbb{C}_0$.



Let $x \in (0, \infty)$. Then as $z \to x$ from the upper half plane,

$$z^\alpha = e^{\alpha(\ln|z| + i\theta)} \to e^{\alpha \ln x} = x^\alpha.$$

As $z \to x$ from the lower plane,

$$z^\alpha = e^{\alpha(\ln|z| + i\theta)} \to e^{\alpha(\ln x + i2\pi)} = x^\alpha e^{2\pi\alpha i}.$$

So z^α behaves differently on the upper edge and the lower edge of the cut. It has to be considered when we compute integrals on branch cuts.

<u>Example</u>: Compute $I = \int_0^\infty \frac{1}{\sqrt{x}\,(x+4)}\,dx$.

<u>Solution</u>: We need to compute

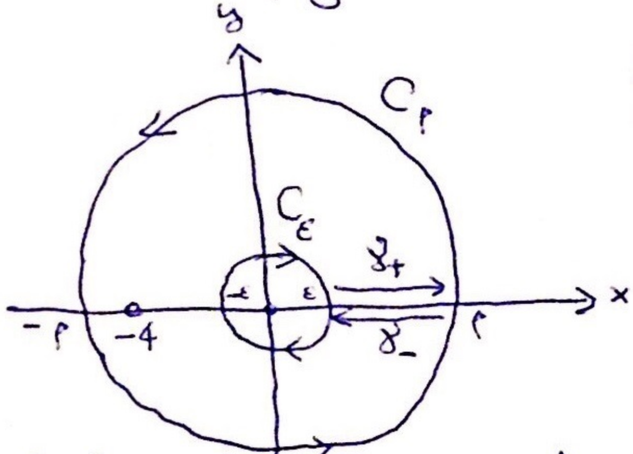
$$\lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \int_\varepsilon^\rho \frac{1}{\sqrt{x}\,(x+4)}\,dx.$$

Let

$$f(z) = \frac{1}{\sqrt{z}\,(z+4)},$$

where

$$\sqrt{z} = e^{\frac{1}{2}\log_0 z} = e^{\frac{1}{2}(\ln|z| + i\theta)}, \quad 0 < \theta \le 2\pi.$$



f is holomorphic on and inside $\Gamma_{\rho,\varepsilon}$ except at -4, which is a simple pole.

$\therefore$ by the Residue Theorem of Cauchy,

$$\int_{C_\rho} f(z)\,dz + \int_{C_\varepsilon} f(z)\,dz + \int_{\gamma_+} f(z)\,dz + \int_{\gamma_-} f(z)\,dz = 2\pi i \operatorname{Res}(f,-4).$$

On γ_+, $\sqrt{z} = \sqrt{x}$; on γ_-, $\sqrt{z} = -\sqrt{x}$.

$$\therefore \int_{\gamma_+} f(z)\,dz + \int_{\gamma_-} f(z)\,dz = \int_\varepsilon^\rho \frac{1}{\sqrt{x}\,(x+4)}\,dx - \int_\varepsilon^\rho \frac{-1}{\sqrt{x}\,(x+4)}\,dx$$

$$= 2\int_\varepsilon^\rho \frac{1}{\sqrt{x}\,(x+4)}\,dx.$$

Now,

$$\left| \int_{C_\rho} f(z)\,dz \right| \leq \frac{2\pi\rho}{\sqrt{\rho}\,(\rho-4)} \to 0$$

as $\rho \to \infty$. Also,

$$\left| \int_{C_\varepsilon} f(z)\,dz \right| \leq \frac{2\pi\varepsilon}{\sqrt{\varepsilon}\,(4-\varepsilon)} \to 0$$

as $\varepsilon \to 0$.

$$\therefore \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \left(\int_{\varepsilon}^{\rho} \frac{1}{\sqrt{x}\,(x+4)}\,dx \right) = 2I$$

$$\therefore 2I = 2\pi i \operatorname{Res}(f,-4)$$

$$= 2\pi i \lim_{z\to -4} \frac{1}{\sqrt{z}} = 2\pi i \lim_{z\to -4} \frac{1}{e^{\frac{1}{2}(\ln|z| + i\pi)}} = -\frac{i}{2}(2\pi i)$$

$$\therefore I = \frac{\pi}{2}.$$