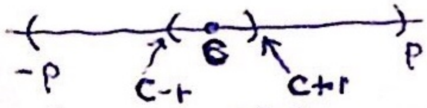
Lecture 8

Singular integrals on $(-\infty,\infty)$ are Cauchy principal values of improper integrals $\int_{-\infty}^{\infty} f(x)\,dx$, where f has a finite number of local singularities on $(-\infty,\infty)$.

Let $f:(-\infty,\infty)\to\mathbb{C}$ be a continuous function except at $c\in(-\infty,\infty)$. Then we define $pv\int_{-\infty}^{\infty} f(x)\,dx$ by



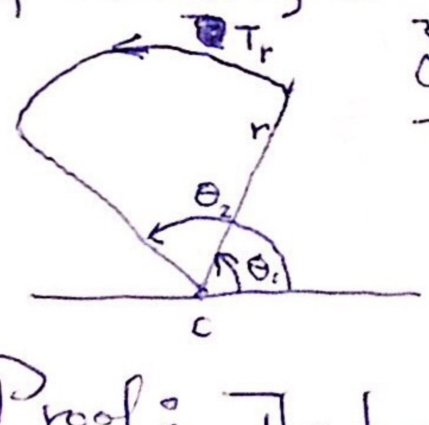
$$pv\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{p\to\infty \\ r\to 0}} \left\{ \int_{-p}^{c-r} f(x)\,dx + \int_{c+r}^{p} f(x)\,dx \right\},$$

provided that the limit exists.

We need a lemma.

Lemma: Let c be a simple pole of a complex-valued function f. Let T_r be the circular arc with parametrization

$$z = c + re^{i\theta}, \quad \theta_1 \le \theta \le \theta_2 .$$



Then

$$\lim_{r\to 0+} \int_{T_r} f(z)\,dz = i(\theta_2-\theta_1)\operatorname{Res}(f,c).$$

Proof: The Laurent series of f at c is given by

$$f(z) = \frac{a_{-1}}{z-c} + \sum_{n=0}^{\infty} a_n (z-c)^n, \quad 0<|z-c|<R,$$

where R is some positive number. Then for $r\in(0,R)$,

$$\int_{T_r} f(z)\,dz = \int_{T_r} \frac{a_{-1}}{z-c}\,dz + \int_{T_r} g(z)\,dz,$$

where

$$g(z) = \sum_{n=0}^{\infty} a_n (z-c)^n.$$

Now, g is holomorphic at $c \Rightarrow g$ is bounded on a neighborhood of c. $\therefore$ there exist positive constants M and R_1 such that

$$|g(z)| \leq M, \quad |z-c| < R_1.$$

So,

$$\left| \int_{T_r} g(z)\,dz \right| \leq M(\theta_2 - \theta_1) r \to 0$$

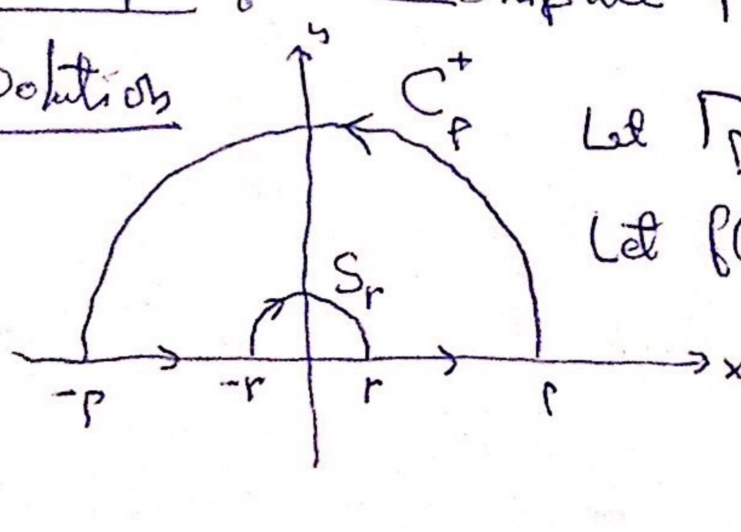
as $r \to 0+$. Also,

$$\int_{T_r} \frac{1}{z-c}\,dz = \int_{\theta_1}^{\theta_2} \frac{1}{re^{i\theta}}\, rie^{i\theta}\,d\theta = i(\theta_2 - \theta_1).$$

$$\therefore \lim_{r \to 0+} \int_{T_r} f(z)\,dz = i(\theta_2 - \theta_1) a_{-1} = i(\theta_2 - \theta_1)\,\mathrm{Res}(f, c)$$

<u>Example:</u> Compute $\mathrm{pv}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$.

<u>Solution</u>



Let $\Gamma_{\rho,r} = C_\rho^+ + [-\rho, -r] + S_r + [r, \rho]$.

Let $f(z) = \dfrac{e^{iz}}{z}$.

Then f has a simple pole at 0, which lies outside $\Gamma_{\rho,r}$. $\therefore$ by Cauchy's integral theorem,

$$0 = \int_{\Gamma_{\rho,r}} \frac{e^{iz}}{z}\,dz$$

$$= \int_{C_\rho^+} \frac{e^{iz}}{z}\,dz + \int_{-\rho}^{-r} \frac{e^{ix}}{x}\,dx + \int_{r}^{\rho} \frac{e^{ix}}{x}\,dx + \int_{S_r} \frac{e^{iz}}{z}\,dz.$$

By Jordan's lemma, $\lim_{\rho\to\infty} \int_{C_\rho^+} \frac{e^{iz}}{z}\,dz = 0$. By the lemma of today,

$$\lim_{r\to 0+} \int_{S_r} \frac{e^{iz}}{z}\,dz = -i(\pi - 0)\,\mathrm{Res}(f,0)$$
$$= -i\pi \lim_{z\to 0} e^{iz} = -i\pi.$$

So,

$$0 = \lim_{\substack{\rho\to\infty \\ r\to 0}} \left\{ \int_{-\rho}^{-r} \frac{e^{ix}}{x}\,dx + \int_{r}^{\rho} \frac{e^{ix}}{x}\,dx \right\} - \pi i$$

$$\therefore \; pv\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i.$$

<u>Remarks</u>: (1) Let $f(x) = \frac{1}{x}$. Then

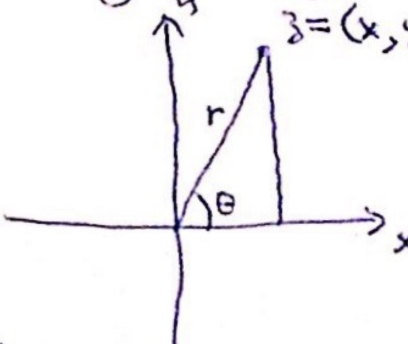
$$\hat{f}(-1) = \frac{1}{\sqrt{2\pi}}\, pv\int_{-\infty}^{\infty} e^{-ix(-1)} \frac{1}{x}\,dx = \frac{1}{\sqrt{2\pi}}\, pv\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\pi i = \sqrt{\frac{\pi}{2}}\, i.$$

(2) $\frac{1}{x}$ is the kernel of the so-called Hilbert transform.

(3) $pv\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \mathrm{Im}\left(pv\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx \right) = \mathrm{Im}(\pi i) = \pi.$

Arguments of Complex Numbers (Chapter 2)

Let $z = x + iy$ be a nonzero complex number. Then



$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

So, z can be represented by (r,θ).

Note that if $z = (r,\theta)$, then $z = (r, \theta + 2k\pi)$, $k \in \mathbb{Z}$.
We call $\theta + 2k\pi$ <u>an</u> argument of z and denote it by $\arg z$. $\therefore$

$$\arg z = \theta + 2k\pi, \quad k \in \mathbb{Z}.$$

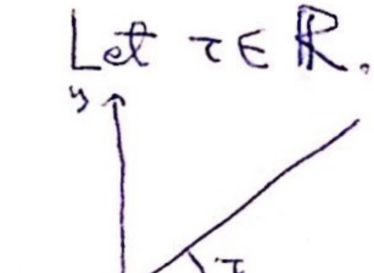
So, if a value (or branch) of $\arg z$ can be found, then so are all values of $\arg z$.

Let $z \in \mathbb{C} - \{0\}$. Let θ be a branch of $\arg z$. Then

$$\arg z = \theta + 2k\pi, \; k \in \mathbb{Z}$$

So, arg is <u>not</u> a function on $\mathbb{C} - \{0\}$.

Let $\tau \in \mathbb{R}$. Then we define $\mathbb{C}_\tau$ by



$$\mathbb{C}_\tau = \{(r,\theta) : r > 0, \; \tau < \theta \le \tau + 2\pi\}.$$

We call it the cut plane along the cut $\{(r,\tau) : r > 0\}$.

Now, we denote the branch of $\arg z$ lying in $(\tau, \tau + 2\pi]$ by $\arg_\tau z$. The most common branch of $\arg z$ is $\arg_{-\pi} z$. It's called the principal branch of $\arg z$ and is denoted by $\operatorname{Arg} z$.