Lecture 7

Fourier Transforms

Let $f:(-\infty,\infty)\to\mathbb{C}$ be a continuous function. Then we define the function $\hat{f}$ on $(-\infty,\infty)$ by

$$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\,\mathrm{pv}\int_{-\infty}^{\infty}e^{-ix\xi}f(x)\,dx,\quad \xi\in(-\infty,\infty),$$

provided that the Cauchy principal values exist. We call $\hat{f}$ the <u>Fourier transform</u> of f.

Two closely related transforms of f are the cosine transform of f and the sine transform of f given by, respectively,

$$\frac{1}{\sqrt{2\pi}}\,\mathrm{pv}\int_{-\infty}^{\infty}\cos(x\xi)f(x)\,dx,$$

$$\frac{1}{\sqrt{2\pi}}\,\mathrm{pv}\int_{-\infty}^{\infty}\sin(x\xi)f(x)\,dx.$$

<u>Example</u>: Let $f(x)=\dfrac{x}{1+x^2}$. Compute $\hat{f}(\xi)$ for all $\xi\in(-\infty,\infty)$.

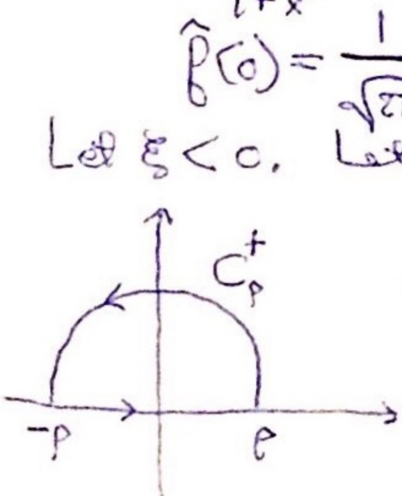
<u>Solution</u>: Let $\xi=0$. Then

$$\hat{f}(0)=\frac{1}{\sqrt{2\pi}}\,\mathrm{pv}\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dx=\frac{1}{\sqrt{2\pi}}\lim_{\rho\to\infty}\int_{-\rho}^{\rho}\frac{x}{1+x^2}\,dx.$$

Since $\dfrac{x}{1+x^2}$ is an odd function of x,

$$\hat{f}(0)=\frac{1}{\sqrt{2\pi}}\lim_{\rho\to\infty}\int_{-\rho}^{\rho}\frac{x}{1+x^2}\,dx=0.$$

Let $\xi<0$. Let $\Gamma_\rho^+=C_\rho^+ + [-\rho,\rho]$. Then



$$\int_{\Gamma_\rho^+}e^{-iz\xi}\frac{z}{1+z^2}\,dz=\int_{C_\rho^+}e^{-iz\xi}\frac{z}{1+z^2}\,dz+\int_{-\rho}^{\rho}e^{-ix\xi}\frac{x}{1+x^2}\,dx$$

Now,

$$\int_{\Gamma_\rho^+} e^{-iz\xi}\frac{z}{1+z^2}\,dz = \int_{\Gamma_\rho^+} e^{-iz\xi}\frac{z}{(z+i)(z-i)}\,dz.$$

Let $f(z) = e^{-iz\xi}\dfrac{z}{(z+i)(z-i)}$. Then f has simple poles at $i, -i$, and i is inside Γ_ρ^+. $\therefore$ By Cauchy's residue theorem,

$$\int_{\Gamma_\rho^+} e^{-iz\xi}\frac{z}{1+z^2}\,dz = 2\pi i \,\mathrm{Res}(f, i)$$
$$= 2\pi i \lim_{z\to i}(z-i)f(z) = 2\pi i \lim_{z\to i}\left(e^{-iz\xi}\frac{z}{z+i}\right)$$
$$= 2\pi i\, e^{\xi}\frac{i}{2i} = \pi i e^{\xi}.$$

$$\therefore \pi i e^{\xi} = \int_{C_\rho^+} e^{-iz\xi}\frac{z}{1+z^2}\,dz + \int_{-\rho}^{\rho} e^{-ix\xi}\frac{x}{1+x^2}\,dx.$$

By Jordan's lemma,

$$\lim_{\rho\to\infty}\int_{C_\rho^+} e^{-iz\xi}\frac{z}{1+z^2}\,dz = 0$$

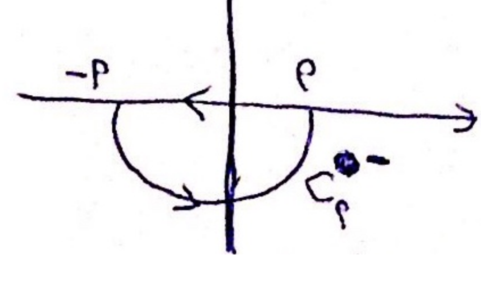
because $\deg(1+z^2) - \deg(z) = 2-1 = 1$. So

$$\mathrm{pv}\int_{-\infty}^{\infty} e^{-ix\xi}\frac{x}{1+x^2}\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} e^{-ix\xi}\frac{x}{1+x^2}\,dx = \pi i e^{\xi}.$$

$$\therefore \frac{1}{\sqrt{2\pi}}\,\mathrm{pv}\int_{-\infty}^{\infty} e^{-ix\xi}\frac{x}{1+x^2}\,dx = \sqrt{\frac{\pi}{2}}\,i e^{\xi}.$$

$$\therefore \hat{f}(\xi) = \sqrt{\frac{\pi}{2}}\,i e^{\xi}, \quad \xi < 0.$$

Now, let $\xi > 0$. Let $\Gamma_\rho^- = C_\rho^- - [-\rho, \rho]$.



Then

$$\int_{\Gamma_\rho^-} e^{-iz\xi}\frac{z}{1+z^2}\,dz = \int_{C_\rho^-} e^{-iz\xi}\frac{z}{1+z^2}\,dz - \int_{-\rho}^{\rho} e^{-ix\xi}\frac{x}{1+x^2}\,dx.$$

Now,

$$\int_{\Gamma_\rho^-} e^{-iz\xi}\frac{z}{1+z^2}\,dz = \int_{\Gamma_\rho^-} e^{-iz\xi}\frac{z}{(z+i)(z-i)}\,dz.$$

Let $f(z) = e^{-iz\xi}\frac{z}{(z+i)(z-i)}$. Then f has simple poles at i and $-i$ and only $-i$ is inside Γ_ρ^-. So

$$\int_{\Gamma_\rho^-} e^{-iz\xi}\frac{z}{1+z^2}\,dz = 2\pi i \operatorname{Res}(f,-i)$$

$$= 2\pi i \lim_{z\to -i}(z+i)f(z)$$

$$= 2\pi i \lim_{z\to -i}\frac{e^{-iz\xi}\,z}{z-i} = 2\pi i\frac{e^{-\xi}(-i)}{-2i}$$

$$= -\pi i e^{-\xi}.$$

Let $\rho\to\infty$ in

$$-\pi i e^{-\xi} = \int_{C_\rho^-} e^{-iz\xi}\frac{z}{1+z^2}\,dz + \int_{-\rho}^{\rho} e^{-ix\xi}\frac{x}{1+x^2}\,dx,$$

we get by Jordan's lemma again,

$$-\pi i e^{-\xi} = pv\int_{-\infty}^{\infty} e^{-ix\xi}\frac{x}{1+x^2}\,dx$$

$$\text{So } \frac{1}{\sqrt{2\pi}}\,pv\int_{-\infty}^{\infty} e^{-ix\xi}\frac{x}{1+x^2}\,dx = -\sqrt{\frac{\pi}{2}}\,i e^{-\xi}$$

So $\hat{f}(\xi) = -\sqrt{\frac{\pi}{2}}\,i e^{-\xi}$ if $\xi > 0$.

To put the answer in an elegant form, we introduce 7.4
a notation. Let $\xi \in \mathbb{R}$. We define $\operatorname{sgn}(\xi)$ (sgn for sign) by

$$\operatorname{sgn}(\xi) = \begin{cases} 1 & \text{if } \xi > 0, \\ 0 & \text{if } \xi = 0, \\ -1 & \text{if } \xi < 0. \end{cases}$$

Then

$$\hat{f}(\xi) = \begin{cases} -\sqrt{\frac{\pi}{2}}\, i e^{-\xi}, & \xi > 0, \\ 0, & \xi = 0, \\ \sqrt{\frac{\pi}{2}}\, i e^{\xi}, & \xi < 0, \end{cases}$$

or succinctly,

$$\hat{f}(\xi) = -\operatorname{sgn}(\xi) \sqrt{\frac{\pi}{2}}\, i e^{-\operatorname{sgn}(\xi)\xi}, \quad \xi \in (-\infty, \infty).$$