Lecture 4

Recall: Let f be a holomorphic function on and inside a simple closed contour Γ. Then

$$\int f(z)\,dz = 0. \quad \text{(Cauchy's integral theorem)}$$

Questions: What happens if f has an isolated singularities inside Γ?

Definition: Let z_0 be an isolated singularity of f. Then we can find a positive number R such that f is holomorphic on $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. The Laurent series of f is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \quad 0 < |z - z_0| < R.$$

The coefficient a_{-1} is called the residue of f at z_0 and is denoted by $\mathrm{Res}(f, z_0)$.

Some examples first

Example 1: Let z_0 be a removable singularity of f. What is $\mathrm{Res}(f, z_0)$?

Solution Write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \quad 0 < |z - z_0| < R.$$

Then $a_{-n} = 0$, $n = 1, 2, \dots$. In particular, $a_{-1} = 0$.

$\therefore \mathrm{Res}(f, z_0) = a_{-1} = 0$

Example 2

Let z_0 be a simple pole of f. Find $\mathrm{Res}(f, z_0)$.

Solution Write f as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + a_{-1}(z-z_0)^{-1}, \quad 0<|z-z_0|<R.$$

So,

$$(z-z_0)f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+1} + a_{-1}, \quad 0<|z-z_0|<R.$$

$$\therefore \mathrm{Res}(f, z_0) = a_{-1} = \lim_{z\to z_0} (z-z_0) f(z).$$

Example 3 Let z_0 be a pole of order $m>1$ of f. Find $\mathrm{Res}(f, z_0)$.

Solution:

Write f as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} a_{-n} (z-z_0)^{-n}, \quad 0<|z-z_0|<R$$

Then $f(z)$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + a_{-1}(z-z_0)^{-1} + \sum_{n=2}^{m} a_{-n}(z-z_0)^{-n},$$

$$0<|z-z_0|<R$$

$\therefore$ $(z-z_0)^m f(z)$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + a_{-1}(z-z_0)^{m-1} + \sum_{n=2}^{m} a_{-n}(z-z_0)^{m-n},$$

$0<|z-z_0|<R$.

$$\left(\frac{d}{dx}\right)^{\alpha}\left(x^{\beta}\right) = \begin{cases} 0 & \text{if } \beta<\alpha, \\ \beta(\beta-1)\cdots(\beta-\alpha+1)\,x^{\beta-\alpha} & \text{if } \beta\geq\alpha, \end{cases}$$

$$= \begin{cases} 0, & \text{if } \beta<\alpha \\ \binom{\beta}{\alpha}\alpha!\,x^{\beta-\alpha}, & \text{if } \beta\geq\alpha \end{cases}$$

$\forall \alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}\cup\{0\}$.

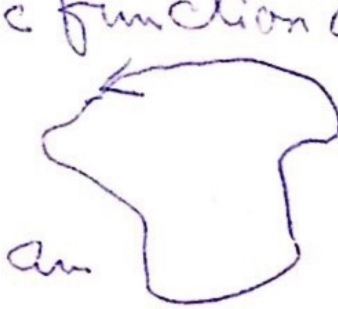
$\therefore$ $\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$

$$= \sum_{n=0}^{\infty} a_n \binom{m+n}{m-1}(m-1)!\,(z-z_0)^{n+1} + a_{-1}(m-1)!\,.$$

$$\therefore \operatorname{Res}(f,z_0) = \lim_{z\to z_0}\frac{1}{(m-1)!}\,\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$$

Recall: Let f be a holomorphic function on a simple closed contour Γ. Then

$$\int_{\Gamma} f(z)\,dz = 0.$$



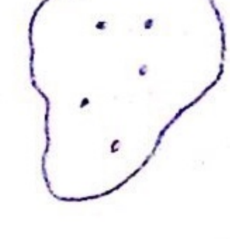
Γ

Question: What happens if f has an isolated singularity inside Γ?

Theorem (Cauchy's Residue Theorem)

Let Γ be a simple closed contour oriented once in the counterclockwise direction. Let f be a holomorphic function on and inside Γ except possibly at isolated singularities $z_1, \dots, z_N$ inside Γ. Then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f, z_j).$$



Proof ($N=2$ only)

The Laurent series of f at z_1 is

$$\sum_{n=0}^{\infty} a_n (z-z_1)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_1)^{-n}, \quad 0<|z-z_1|<c_1$$

Cauchy's integral theorem

$$\Longrightarrow \int_\Gamma f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$\int_{C_1} f(z)\,dz = a_{-1} = \operatorname{Res}(f, z_1),$$ Similarly,

$$\int_{C_2} f(z)\,dz = \operatorname{Res}(f, z_2)$$

$$\therefore \int_\Gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right).$$