Lecture 12

To study $\mathbb{D}$ and $\mathbb{H}$ in depth, we need some preparation.

Zeros of Holomorphic Functions

Theorem: Let f be a holomorphic function on a domain D in $\mathbb{C}$. If $Z = \{z \in D : f(z) = 0\}$ has a limit point in D, then $f(z) = 0$ for all $z \in D$.

Comments: ① We call Z the zero set of f.

②

Let $z^* \in D$. Then z^* is a limit point of Z if there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in Z such that $z_n \neq z^*$ and

$$\lim_{n\to\infty} z_n = z^*.$$

③ If f and g are holomorphic on D and $f = g$ on a set Z with a limit point in D, then $f(z) = g(z)$ for all $z \in D$.

④ We call the theorem the Unique Continuation Property.

Lemma: Let z^* be a limit point of Z in the theorem. Then $f^{(n)}(z^*) = 0$ for $n = 0, 1, 2, \ldots$. In other words,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z^*)}{n!}(z - z^*)^n = 0$$

for all z in a neighborhood of z^*.

Proof: $f(z^*) = 0$. Indeed, let $\{z_n\}_{n=1}^{\infty}$ be a sequence in Z such that $z_n \neq z^*$ and $\lim_{n\to\infty} z_n = z^*$. Then

$$f(z^*) = \lim_{n\to\infty} f(z_n) = 0$$

Now, suppose that there exists a positive integer n

with $f^{(n)}(z^*) \neq 0$. Let k be the smallest positive integer with $f^{(k)}(z^*) \neq 0$. Then there exists a positive number R such that for $|z-z^*| < R$,

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z^*)}{n!}(z-z^*)^n = (z-z^*)^k \sum_{n=k}^{\infty} \frac{f^{(n)}(z^*)}{n!}(z-z^*)^{n-k}$$

Let $g(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z^*)}{n!}(z-z^*)^{n-k}$. Then

$$g(z^*) = f^{(k)}(z^*) \neq 0.$$

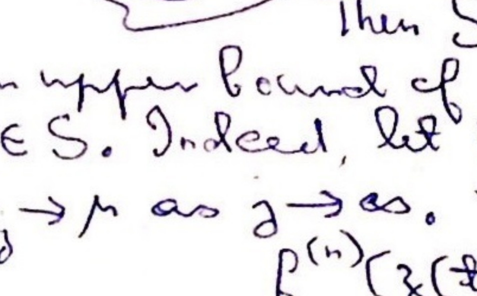
But

$$f(z_n) = (z_n - z^*)^k g(z_n) = 0$$

$$\Rightarrow g(z_n) = 0 \Rightarrow g(z^*) = \lim_{n\to\infty} g(z_n) = 0.$$

This is a contradiction.

Proof of Unique Continuation Property: Let z^* be a limit point of Z in D. Then $f(z)=0$ for all z in a neighborhood of z^*. Let $\zeta \in D$. We join z^* and ζ by a contour lying in D. Parametrizing the contour by



$$z = z(t), \quad a \le t \le b,$$

let $S = \{t \in [a,b] : f \text{ is zero on a neighborhood of } z(t)\}$.

Then $S \neq \emptyset$ because $a \in S$. Also, b is an upper bound of S. $\therefore\ \mu = \sup S$ exists. Now $\mu \in S$. Indeed, let $\{t_j\}_{j=1}^{\infty}$ be a sequence in S with $t_j \to \mu$ as $j \to \infty$. Then for $n = 0, 1, 2, \ldots$,

$$f^{(n)}(z(t_j)) \to f^{(n)}(z(\mu))$$

as $j \to \infty$. $\therefore$

$$f^{(n)}(z(\mu)) = 0, \quad n = 0, 1, 2, \ldots.$$

$\therefore\ \mu \in S$. In fact, $\mu = b$. If not, then $\mu < b$. So, there exists a number $r > 0$ such that

$$f(z) = 0, \quad |z - z(\mu)| < r.$$

Hence there exists a number $\nu \in (\mu, b)$ such that

$$f(z) = 0, \quad |z - \gamma(\nu)| < r_1$$

where r_1 is some positive number $\therefore \nu \in S$ and this contradicts that μ is the least upper bound.

<u>Theorem</u> (Maximum Modulus Principle) Let f be a <u>nonconstant</u> holomorphic function on a domain D of $\mathbb{C}$. Then $|f|$ cannot a local maximum in D.

<u>Proof</u>: Suppose that $|f|$ attains a local maximum at $z_0 \in D$. Then there exists a positive number R such that

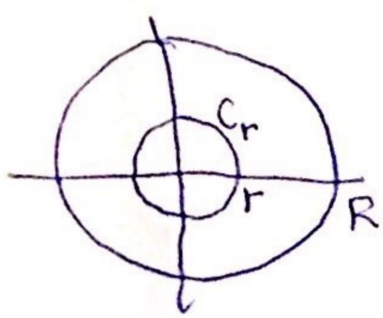
$$|f(z)| \le |f(z_0)|, \quad |z - z_0| < R.$$

If $f(z_0) = 0$, then $f(z) = 0$ for all z with $|z - z_0| < R$. $\therefore$ by the unique continuation property, $f(z) = 0$ for all $z \in D$, a contradiction. So, suppose that $f(z_0) \ne 0$.

Let $\lambda = \dfrac{|f(z_0)|}{f(z_0)}$. Then by the Mean Value Property of holomorphic functions,

$$|f(z_0)| = \lambda f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \lambda f(z_0 + re^{i\theta})\, d\theta,$$

where $0 < r < R$.



$$\therefore |f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\{\lambda f(z_0 + re^{i\theta})\}\, d\theta$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} \{|f(z_0)| - \mathrm{Re}(\lambda f(z_0 + re^{i\theta}))\}\, d\theta = 0.$$

But
$$|f(z_0)| - \mathrm{Re}(\lambda f(z_0+re^{i\theta}))$$
$$\geq |f(z_0)| - |\lambda f(z_0+re^{i\theta})|$$
$$= |f(z_0)| - |f(z_0+re^{i\theta})| \geq 0.$$
$\therefore$ $|f(z_0)| = \mathrm{Re}(\lambda f(z))$, $z \in C_r$.

Also, for all $z \in C_r$,
$$|f(z_0)|^2 \geq |\lambda f(z)|^2$$
$$= (\mathrm{Re}(\lambda f(z)))^2 + (\mathrm{Im}(\lambda f(z)))^2$$
$$= |f(z_0)|^2 + (\mathrm{Im}(\lambda f(z)))^2.$$
$\therefore$ $\mathrm{Im}(\lambda f(z)) = 0$, $z \in C_r$.

$\therefore$ $\lambda f(z) = |f(z_0)|$, $z \in C_r \Rightarrow f(z) =$ constant on C_r.

By the unique continuation property,
$$f(z) = \text{constant}, \ z \in D.$$
This is a contradiction.