Lecture 11

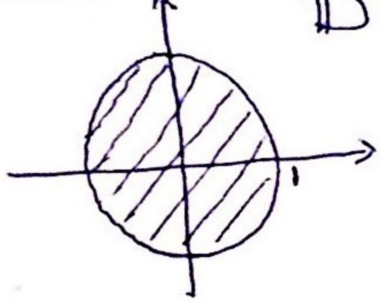
Biholomorphisms

Let D_1 and D_2 be domains in $\mathbb{C}$. Let $f: D_1 \to D_2$ be a bijective holomorphism. Then we say that D_1 and D_2 are biholomorphic or conformly equivalent. We call $f: D_1 \to D_2$ a biholomorphism or a conformal mapping.

Two Canonical Simply Connected Domains

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

The unit disk centered at 0



$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

The upper half plane

Two Canonical Mappings

Let $F : \mathbb{C} - \{-i\} \to \mathbb{C}$ be defined by

$$F(z) = \frac{i-z}{i+z}, \quad z \in \mathbb{C} - \{-i\}.$$

Let $G : \mathbb{C} - \{-1\} \to \mathbb{C}$ be defined by

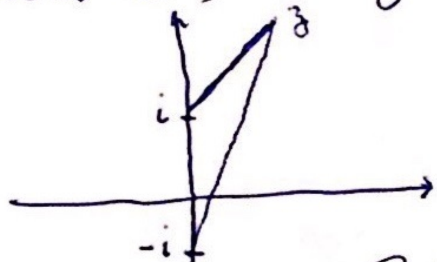
$$G(\omega) = i\,\frac{1-\omega}{1+\omega}, \quad \omega \in \mathbb{C} - \{-1\}.$$

Theorem: $F : \mathbb{H} \to \mathbb{D}$ is a bihomorphism with inverse $G : \mathbb{D} \to \mathbb{H}$.

Proof: $F : \mathbb{H} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{H}$ are obviously holomorphic. To see that F maps $\mathbb{H}$ into $\mathbb{D}$, let $z \in \mathbb{H}$, then

$$|F(z)| = \left|\frac{i-z}{i+z}\right| < 1$$

$$\Rightarrow F(z) \in \mathbb{D}.$$



To see that $G : \mathbb{D} \to \mathbb{H}$, let $\omega \in \mathbb{D}$, then

$$G(\omega) = i\,\frac{1-\omega}{1+\omega}.$$

Write $\omega = u + iv$. Then

$$G(\omega) = i\,\frac{1-u-iv}{1+u+iv}$$

$$= \frac{v+(1-u)i}{(1+u)+iv} = \frac{(v+(1-u)i)((1+u)-iv)}{(1+u)^2+v^2}$$

$$= \frac{2(1+u)v + i(1-u^2-v^2)}{(1+u)^2+v^2}.$$

$$\therefore \operatorname{Im}(G(\omega)) = \frac{1-(u^2+v^2)}{\left[1+u^2\right)+v^2} > 0.$$

$\therefore G(\omega) \in \mathbb{H}$.

Now, let $\omega \in \mathbb{D}$. Then

$$F(G(\omega)) = \frac{i - G(\omega)}{i + G(\omega)}$$

$$= \frac{i - i\,\frac{1-\omega}{1+\omega}}{i + i\,\frac{1-\omega}{1+\omega}} = \frac{i + i\omega - i + i\omega}{i + i\omega + i - i\omega} = \frac{2i\omega}{2i}$$

$$= \omega.$$

$\forall z \in \mathbb{H}$,

$$G(F(z)) = i\,\frac{1-F(z)}{1+F(z)}$$

$$= i\,\frac{1-\frac{i-z}{1+z}}{1+\frac{i-z}{1+z}} = i\,\frac{1+z-i+z}{1+z+i-z}$$

$$= i\,\frac{2z}{2i} = z.$$

$\therefore FG = GF = I.$

<u>Remarks</u> ① $F : \mathbb{H} \to \mathbb{D}$ given by

$$F(z) = \frac{i-z}{i+z},\ z \in \mathbb{H},$$

is called the <u>Cayley transform</u>. It is a very special case of fractional linear transformations of the form

$$\omega = \frac{az+b}{cz+d},\ z \neq -\frac{d}{c},$$

where

① a, b, c, d are complex numbers such that the denominator is not a multiple of the numerator.

② $c \neq 0$. If $c = 0$, then ω is of the form

$$\omega = a'z + b'.$$

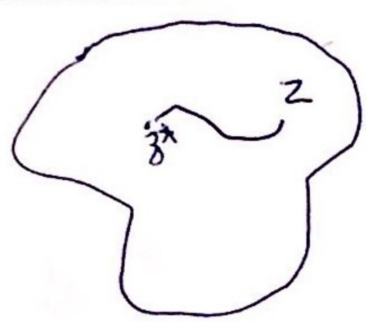
③ The Cayley transform is the biholomorphism giving the conformal equivalence of $\mathbb{H}$ and $\mathbb{D}$. This is the standard example of the Riemann mapping theorem stating that if D is a simply connected domain in $\mathbb{C}$ with $D \neq \mathbb{C}$, then D is biholomorphic to $\mathbb{D}$.

To study $\mathbb{H}$ and $\mathbb{D}$ in detail, we need

Zeros of Holomorphic functions

Theorem Let Z be the zero set of a holomorphic function f on a domain D. If Z has a limit point in D, then f is identically zero on D.

Remarks: ① What is a limit point of Z in D?



Let z^* be a limit point of Z in D. Then $\forall$ ε-disk centered and punctured at z^*, i.e., $D(z^*,\varepsilon)-\{z^*\}$, $\exists\, z \in Z$ $\ni z \in D(z^*,\varepsilon)-\{z^*\}$.

② The theorem says that if f vanishes on a very tiny set with a limit point, then f vanishes everywhere.

③ If f and g are holomorphic on a domain D and $f = g$ on a very tiny set with a limit point in D, then $f(z) = g(z)$ $\forall z \in D$.

④ Property ③ is known as the Unique Continuation Property of Holomorphic Functions.