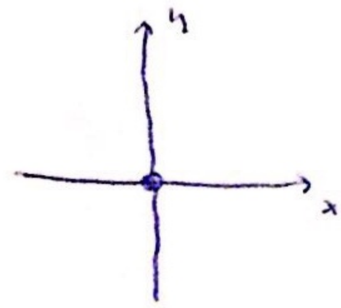
Lecture 3



Example Find the Laurent series of $e^{\frac{1}{z}}$.

Solution $e^{\frac{1}{z}}$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

Its Laurent series is

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} z^{-n}, \quad z \neq 0.$$

Question: Why Laurent series?

Definition: Let $\underset{\substack{\cap\!| \text{ open}\\ \mathbb{C}}}{G} \to \mathbb{C}$ be a function such that

- f is not holomorphic at a point $z_0 \in G$,
- f is holomorphic on $\{z \in G : 0 < |z - z_0| < R\} \subseteq G$, where $R > 0$.

Then we call z_0 an <u>isolated singularity</u> of f.

Let z_0 be an isolated singularity of f. Then f has a Laurent series expansion as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \quad 0 < |z - z_0| < R.$$

Case 1: $a_{-n} = 0$ for all $n = 1, 2, \cdots$. Then we call z_0 a <u>removable singularity</u> of f.

<u>Case 2</u>: If $a_{-m} \neq 0$ and $a_{-n} = 0$ for all $n > m$, then we call z_0 <u>a pole of order m</u>. If $m=1$, z_0 is called a <u>simple pole</u>.

<u>Case 3</u>: If $a_{-n} \neq 0$ for infinitely many positive integers n, then z_0 is an <u>essential singularity</u>.

<u>Examples</u> 1. Let $f(z) = e^{\frac{1}{z}}$. Find all isolated singularities and classify each isolated singularity.

<u>Solution</u>: We have seen, 0 is an isolated singularity and the only one. $\therefore e^{\frac{1}{z}} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} z^{-n}$. Since all $a_{-n} \neq 0$, $\therefore$ 0 is an essential singularity.

<u>Example 2</u>. Let $f(z) = \frac{\sin z}{z}$. Find all isolated singularities and classify each isolated singularity.

<u>Solution</u>: 0 is the only isolated singularity.

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right) = \text{the only holomorphic part.}$$

$\therefore$ 0 is a removable singularity.

<u>Example 3</u>. Let $f(z) = \frac{e^z}{z^2}$. Find all isolated singularities and classify each isolated singularity.

<u>Solution</u>: 0 is the only isolated singularity.

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + z + \frac{z^2}{2!} + \cdots\right) = \left(\frac{1}{2!} + \frac{1}{3!} z + \cdots\right) + \left(z^{-1} + z^{-2}\right).$$

$\therefore z = 0$ is a pole of order 2.