Lecture 2

Let $I_2 = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta$. We get

$$I_2 = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \text{ where}$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta, \quad n = 0, 1, 2, \dots \qquad 2.1$$

Now, let $I_1 = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta$. Then

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = -\frac{1}{z - z_0}\,\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}}.$$

$$= -\frac{1}{z-z_0}\sum_{j=0}^{n} \frac{(\zeta - z_0)^j}{(z - z_0)^j} - \frac{1}{z - z_0}\,\frac{\frac{(\zeta - z_0)^{n+1}}{(z-z_0)^{n+1}}}{1 - \frac{\zeta - z_0}{z - z_0}}$$

$$= -\sum_{j=0}^{n} \frac{(\zeta - z_0)^j}{(z - z_0)^{j+1}} - \frac{1}{z - \zeta}\,\frac{(\zeta - z_0)^{n+1}}{(z - z_0)^{n+1}}$$

$$\therefore I_1 = -\sum_{j=0}^{n} (z - z_0)^{-j-1} \int_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-j}}\,d\zeta$$

$$- \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{z - \zeta}\,\frac{(\zeta - z_0)^{n+1}}{(z - z_0)^{n+1}}\,d\zeta$$

$$= -\sum_{j=1}^{n+1} \int_{C_1} \frac{f(\zeta)(z - z_0)^{-n}}{(\zeta - z_0)^{-j+1}}\,d\zeta + \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\,\frac{(\zeta - z_0)^{n+1}}{(z - z_0)^{n+1}}\,d\zeta.$$

But $\left|\dfrac{\zeta - z_0}{z - z_0}\right|^{n+1} \le \left(\dfrac{R_1}{\rho'}\right)^{n+1} \to 0$ as $n \to \infty$.

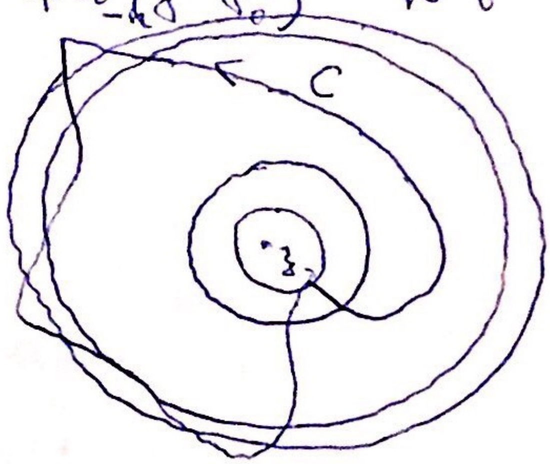
$$|\zeta - z| \le \frac{1}{R_1 - r_1},$$

$\therefore \left|\dfrac{f(\zeta)}{\zeta - z}\right| \le \max\limits_{\zeta \in C_1} |f(\zeta)| \dfrac{1}{R_1 - r_1}$

$\therefore \left|\dfrac{1}{2\pi}\displaystyle\int_{C_1} \dfrac{f(\zeta)}{\zeta - z}\dfrac{(\zeta - z_0)^{n+1}}{(z - z_0)^{n+1}}\, d\zeta\right| \to 0$ as $n \to \infty$

So $I_1 = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}} (z - z_0)^n$.

So $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n + a_{-n}(z - z_0)^{-n}$ To find a formula for a_n, $n \in \mathbb{Z}$



Now $C_2 \to C$,

$C_1 \to C$.

So

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

Theorem: Let $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\displaystyle\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ be series with

- $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$ Converges on $\{z \in \mathbb{C} : |z - z_0| < R\}$.
- $\displaystyle\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ Converges on $\{z \in \mathbb{C} : |z - z_0| > r\}$.
- $r < R$.

Then there exists a unique holomorphic function f

on $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ with Laurent series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}, \quad r < |z - z_0| < R.$$

<u>Proof</u> ($z_0 = 0$)

Let $\zeta = \frac{1}{z}$. Then $\sum_{n=1}^{\infty} a_{-n} z^{-n} = \sum_{n=1}^{\infty} a_{-n} \zeta^{-n}$ converges on $\{\zeta \in \mathbb{C} : |\zeta| < \frac{1}{r}\}$. $\therefore$ $H(\zeta) = \sum_{n=1}^{\infty} a_{-n}\zeta^{-n}$ is holomorphic on $\{\zeta \in \mathbb{C} : |\zeta| < \frac{1}{r}\}$.

Now $h(z) = H\left(\frac{1}{z}\right)$ is holomorphic on $\{z \in \mathbb{C} : |z| > r\}$

$$= \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

But $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on $\{z \in \mathbb{C} : |z| < R\}$.

Let $f(z) = h(z) + g(z)$, $z \in \{z \in \mathbb{C} : r < |z - z_0| < R\}$.

Then f is holomorphic in D.

Let C be any simple closed contour in D enclosing z_0 oriented once in the counterclockwise direction.

Then

$$\frac{1}{2\pi i}\int_C \left(\sum_{n=-\infty}^{\infty} a_n z^n\right) \Big/ z^{m+1}\, dz$$

$$= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} \int_C a_n z^{n-m-1}\, dz$$

$$= \frac{1}{2\pi i}\begin{cases} \frac{1}{2\pi i} a_m, & n = n, \\ 0, & n \neq m. \end{cases}$$

$$= \begin{cases} a_m, & n = m. \\ 0, & n \neq m. \end{cases}$$

$$\therefore\ a_m = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{m+1}}\, dz.$$

Example. Find the Laurent series of

$$f(z) = \frac{z^2-2z+3}{z-2}$$

on $\{ z \in \mathbb{C} : |z-1| > 1 \}$

Solution $$f(z) = \frac{(z-1)^2+2}{(z-1)-1} = \left((z-1)^2+2\right)\frac{1}{z-1}\,\frac{1}{1-\frac{1}{z-1}}$$

$$= \left((z-1) + \frac{2}{z-1}\right) \sum_{n=0}^{\infty} \frac{1}{(z-1)^n}$$

$$= \sum_{n=0}^{\infty} (z-1)^{-n+1} + 2\sum_{n=0}^{\infty} (z-1)^{-n-1}$$

$$= (z-1) + 1 + \sum_{n=2}^{\infty} (z-1)^{-n+1} + 2\sum_{n=0}^{\infty} (z-1)^{-n-1}$$

Put $j = n-1$. Then

$$\sum_{n=2}^{\infty} (z-1)^{-n+1} = \sum_{j=1}^{\infty} (z-1)^{-j}$$

Put $k = n+1$. Then

$$\sum_{n=0}^{\infty} (z-1)^{-(n-1)} = \sum_{k=1}^{\infty} (z-1)^{-k}$$

$$\therefore\ f(z) = (z-1) + 1 + \sum_{n=1}^{\infty} 3(z-1)^{-k}$$