Lecture 1

Let G be an open subset of $\mathbb{C}$. Let $f: G \to \mathbb{C}$. Let $z_0 \in G$. Suppose that f is differentiable at every point in a neighborhood of z_0. Then we say that f is holomorphic at z_0.

Fact: Suppose that f is holomorphic at z_0. Then f' is holomorphic at z_0. More generally, $f^{(n)}$ is holomorphic at z_0 for all $n \in \mathbb{N} \cup \{0\}$.

Definition: Let f be a holomorphic function at z_0. Then $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ is called the Taylor series of f at z_0. If $z_0 = 0$, then $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ is called the Maclaurin series of f.

Theorem Let f be a holomorphic function on the open disk $D = \{z \in \mathbb{C} : |z-z_0| < R\}$, $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \quad z \in D,$$

where the series converges uniformly on every closed subdisk

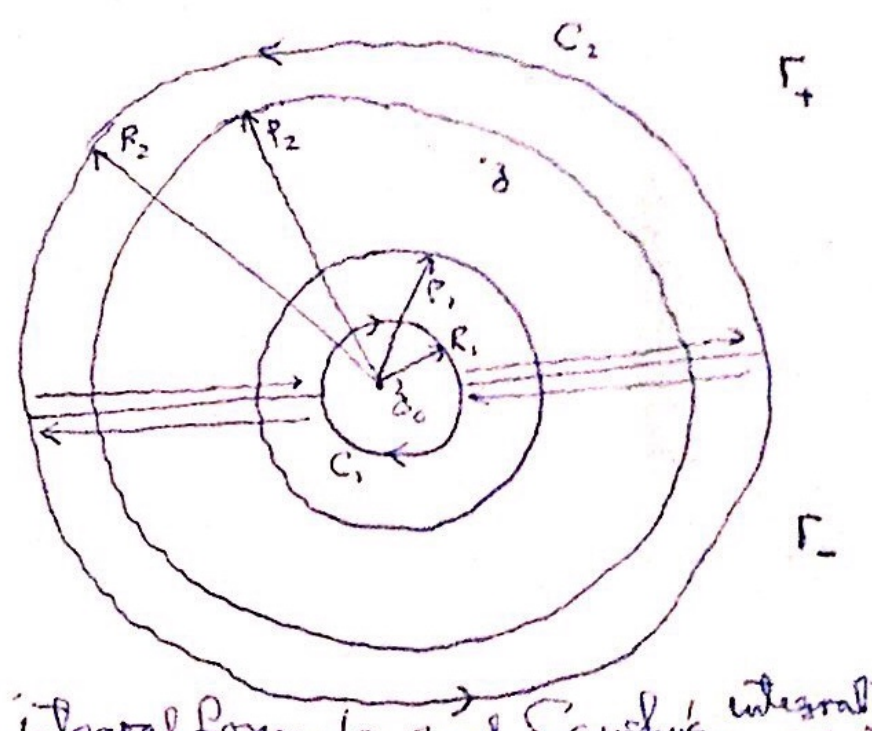
$$\{z \in \mathbb{C} : |z-z_0| \leq \rho\}$$

of D.

Question: What happens if f is holomorphic on an annulus instead of a disk?

Theorem: Let f be a holomorphic function on
$$D = \{ z \in \mathbb{C} : r < |z - z_0| < R \},$$
where $0 \le r < R \le \infty$. Then
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}, \quad z \in D,$$
where both series converge uniformly on every closed subannulus
$$\{ z \in \mathbb{C} : \rho_1 \le |z - z_0| \le \rho_2 \}$$
of D. Moreover,
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta, \quad n \in \mathbb{Z},$$
where C is any simple closed contour in D enclosing z_0 and oriented once in the counterclockwise direction.

Proof



By Cauchy's integral formula and Cauchy's integral theorem,
$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_+} \frac{f(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{\Gamma_-} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

$\therefore$ $$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

For $I_2 = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta$, write

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{(\zeta - z_0)}\,\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

Let $\omega = \frac{z - z_0}{\zeta - z_0}$. Then $\sum_{j=0}^{n} \omega^j = \frac{1 - \omega^{n+1}}{1 - \omega}$.

So,

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{j=0}^{n} (z - z_0)^j \frac{1}{(\zeta - z_0)^j} + \frac{\frac{(z - z_0)^{n+1}}{(\zeta - z_0)^{n+1}}}{1 - \frac{z - z_0}{\zeta - z_0}}$$

$$\therefore \frac{1}{\zeta - z} = \sum_{j=0}^{n} (z - z_0)^j \frac{1}{(\zeta - z_0)^{j+1}} + \frac{(z - z_0)^{n+1}}{(\zeta - z_0)^{n+1}}\,\frac{1}{\zeta - z}$$

So,

$$I_2 = \underbrace{\sum_{j=0}^{n} \left(\frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\,d\zeta\right)(z - z_0)^j}_{\sum_{j=0}^{n} a_j (z - z_0)^j} + \underbrace{\frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\,\frac{(z - z_0)^{n+1}}{(\zeta - z_0)^{n+1}}\,d\zeta}_{T_n(z)}$$

Now

$$\left|\frac{z - z_0}{\zeta - z_0}\right| \le \frac{\rho_2}{R_2}, \quad \frac{1}{|\zeta - z|} \le (R_2 - \rho_2)^{-1}$$

$$\therefore \left|T_n(z)\right| \le \frac{1}{2\pi}\left(\frac{\rho_2}{R_2}\right)^{n+1} \frac{1}{R_2 - \rho_2}\, 2\pi R_2 \longrightarrow 0 \text{ as } n \to \infty$$

$$\therefore \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$