# Solutions to Assigment 1

12.1. We first write for all $z$ with $0 < |z-4| < 4$,

$$\begin{aligned}\frac{z+1}{z(z-4)^3} &= \left(1+\frac{1}{z}\right)\frac{1}{(z-4)^3}\\ &= \left(1+\frac{1}{(z-4+4)}\right)\frac{1}{(z-4)^3}\\ &= \left(1+\frac{1}{4\left(1+\frac{z-4}{4}\right)}\right)\frac{1}{(z-4)^3}.\end{aligned}$$

Using the geometric series expansion, we get

$$\begin{aligned}&\frac{z+1}{z(z-4)^3}\\ =\ &\left(1+\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(z-4)^n}{4^n}\right)\frac{1}{(z-4)^3}\\ =\ &\frac{1}{(z-4)^3}+\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(z-4)^{n-3}\\ =\ &(z-4)^{-3}+\frac{1}{4}(z-4)^{-3}-\frac{1}{16}(z-4)^{-2}+\frac{1}{64}(z-4)^{-1}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{4^{n+4}}(z-4)^n\\ =\ &\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{4^{n+1}}(z-4)^n+\frac{1}{64}(z-4)^{-1}-\frac{1}{16}(z-4)^{-2}+\frac{5}{4}(z-4)^{-3}.\end{aligned}$$

12.2. Using the Taylor series expansion of cos, we get

$$\begin{aligned}z^2\cos\left(\frac{1}{3z}\right) &= z^2\left(1-\frac{1}{2!}\frac{1}{(3z)^2}+\frac{1}{4!}\frac{1}{(3z)^4}-\cdots\right)\\ &= -\frac{1}{18}+z^2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!3^{2n}}z^{-2n}.\end{aligned}$$

12.3. The Laurent series of $e^{1/z}$ at 0 is

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

Let $C$ be the unit circle with center at the origin and oriented once in the counterclockwise direction. Then

$$\frac{1}{2\pi i} \int_C \frac{e^{1/z}}{z^{-n+1}} dz, \quad n = 0, 1, 2, \ldots.$$

Parametrizing $C$ by

$$z = e^{i\theta}, \quad -\pi \le \theta \le \pi,$$

we get for all $n = 0, 1, 2, \ldots,$

$$\begin{aligned}
& \frac{1}{2\pi i} \int_C \frac{e^{1/z}}{z^{-n+1}} dz \\
= & \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{e^{-i\theta}}}{e^{i(-n+1)\theta}} i e^{i\theta} d\theta \\
= & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{e^{-i\theta}}}{e^{in\theta}} d\theta \\
= & \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\cos\theta - i\sin\theta} (\cos(n\theta) + i\sin(n\theta))\, d\theta \\
= & \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\cos\theta} (\cos(\sin\theta) - i\sin(\sin\theta))(\cos(n\theta) + i\sin(n\theta))\, d\theta \\
= & \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\cos\theta} (\cos(\sin\theta)\cos(n\theta) + \sin(\sin\theta)\sin(n\theta))\, d\theta \\
+ & \frac{i}{2\pi} \int_{-\pi}^{\pi} e^{\cos\theta} (\cos(\sin\theta)\sin(n\theta) - \sin(\sin\theta)\cos(n\theta))\, d\theta.
\end{aligned}$$

The integrand in the second last line is an even function of $\theta$ and the integrand in the last line is an odd function of $\theta$. Therefore

$$\frac{1}{2\pi i} \int_C \frac{e^{1/z}}{z^{n+1}} dz = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{\cos\theta} \cos(\sin\theta + n\theta)\, d\theta.$$

So,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{\cos\theta}(\cos(\sin\theta)+n\theta)\,d\theta = \frac{1}{n!}, \quad n = 0, 1, 2, \ldots.$$

12.4. The function $e^{\left[\frac{w}{2}\left(z-\frac{1}{z}\right)\right]}$ has an isolated singularity at 0. Writing the Laurent series of $e^{\left[\frac{w}{2}\left(z-\frac{1}{z}\right)\right]}$ as

$$e^{\left[\frac{w}{2}\left(z-\frac{1}{z}\right)\right]} = \sum_{n=-\infty}^{\infty} J_n(w) z^n, \quad z \in \mathbb{C} - \{0\},$$

we have

$$J_n(w) = \frac{1}{2\pi i}\int_C \frac{e^{\left[\frac{w}{2}\left(z-\frac{1}{z}\right)\right]}}{z^{n+1}} dz, \quad n \in \mathbb{Z}.$$

Let $C$ be the unit circle centered at the origin and oriented once in the counterclockwise direction. Parametrizing $C$ by $z^{i\theta}$ with $0 \le \theta \le 2\pi$, we get

$$\begin{aligned} J_n(w) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\left[\frac{w}{2}(e^{i\theta}-e^{-i\theta})\right]}}{e^{i(n+1)\theta}} ie^{i\theta} d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} e^{iw\sin\theta} ie^{-in\theta} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}e^{iw\sin\theta} d\theta \end{aligned}$$

for all $n \in \mathbb{Z}$. Now, by periodicity,

$$\begin{aligned}
& \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}e^{iw\sin\,\theta}d\theta \\
= & \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}e^{iw\sin\,\theta}d\theta \\
= & \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos(n\theta)-i\sin(n\theta))(\cos(w\sin\,\theta)+i\sin(w\sin\,\theta))d\theta \\
= & \frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(n\theta)\cos(w\sin\,\theta)+\sin(n\theta)\sin(w\sin\,\theta)]\,d\theta \\
+ & \frac{i}{2\pi}\int_{-\pi}^{\pi}[\cos(n\theta)\sin(w\sin\,\theta)-\sin(n\theta)\cos(w\sin\,\theta)]\,d\theta \\
= & \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(w\sin\,\theta-n\theta)d\theta+i\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(w\sin\,\theta-n\theta)\,d\theta
\end{aligned}$$

In the last line, the first integrand is an even function of $\theta$ and the second integrand is an odd function of $\theta$. Therefore

$$J_n(w)=\frac{1}{\pi}\int_0^{\pi}\cos(w\sin\,\theta-n\theta)\,d\theta$$

for all $n \in \mathbb{Z}$.

12.6.(a) Writing

$$\frac{z^5}{z^3+z}=\frac{z^5}{z(z+i)(z-i)},$$

we see that the isolated singularities are 0 and $\pm i$. Since $\frac{z^5}{(z+i)(z-i)}$ is holomorphic on a neighborhood of 0, we can write it as a power series centered at 0. So,

$$\frac{z^5}{(z+i)(z-i)}=a_0+a_1z+a_2z^2+\cdots.$$

Let $z=0$. Then we see that $a_0=0$. Therefore

$$\frac{z^5}{z(z+i)(z-i)}=a_1+a_2z+\cdots.$$

So,

$$a_{-n} = 0, \quad n = 1, 2, \ldots.$$

Therefore 0 is a removable singularity. Similarly, we can show that $\pm i$ are simple poles.

(b) 0 is the only isolated singularity of $z^4 \sin\left(\frac{1}{z^2}\right)$. We have

$$z^4 \sin\left(\frac{1}{z^2}\right) = z^4\left(\frac{1}{z^2} - \frac{1}{3!}\frac{1}{z^6} + \frac{1}{5!}\frac{1}{z^{10}} - \cdots\right) = z^2 - \frac{1}{3!}z^{-2} + \frac{1}{5!}z^{-6} - \cdots.$$

Therefore 0 is an essential singularity.

(c) Since $\frac{\cos z}{z^2-1} = \frac{\cos z}{(z+1)(z-1)}$, the only isolated singularities are 1 and $-1$ and they are simple poles.

13.1.(a) The isolated singularities of $f(z) = \frac{e^z}{z(z+1)^3}$ are 0 and $-1$. 0 is a simple pole and

$$\mathrm{Res}(f, 0) = \lim_{z\to 0}(zf(z)) = \lim_{z\to 0}\frac{e^z}{(z+1)^3} = 1.$$

Now,

$$\mathrm{Res}(f, -1) = \lim_{z\to -1}\frac{1}{2!}\frac{d^2}{dz^2}((z+1)^3 f(z)) = \lim_{z\to -1}\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{e^z}{z}\right).$$

But

$$\frac{d}{dz}\left(\frac{e^z}{z}\right) = \frac{ze^z - e^z}{z^2}$$

and hence

$$\frac{d^2}{dz^2}\left(\frac{e^z}{z}\right) = \frac{z^3e^z - 2z^2e^z + 2ze^z}{z^4}.$$

So,

$$\mathrm{Res}(f, -1) = -\frac{5}{2}e^{-1}.$$

(b). The only isolated singularity of $f(z) = \sin\left(\frac{1}{3z}\right)$ is 0. Since

$$\sin\left(\frac{1}{3z}\right) = \frac{1}{3z} - \frac{1}{3!}\frac{1}{(3z)^3} + \frac{1}{5!}\frac{1}{(3z)^5} - \cdots = \frac{1}{3}z^{-1} - \frac{1}{162}z^{-3} + \frac{1}{29160}z^{-5} - \cdots .$$

Thus,

$$\text{Res}(f, 0) = \frac{1}{3}.$$

13.1.(a) Let $f(z) = \frac{\sin z}{z^2-4} = \frac{\sin z}{(z-2)(z+2)}$. Then $f$ has two simple poles $\pm 2$. They are both inside $C$. By Cauchy's residue theorem,

$$\int_C \frac{\sin z}{z^2 - 4} dz = 2\pi i(\text{Res}(f, 2) + \text{Res}(f, -2)).$$

But

$$\text{Res}(f, 2) = \lim_{z \to 2}(z - 2)f(z) = \lim_{z \to 2} \frac{\sin z}{z + 2} = \frac{\sin 2}{4}$$

and

$$\text{Res}(f, -2) = \lim_{z \to -2}(z + 2)f(z) = \lim_{z \to -2} \frac{\sin z}{z - 2} = \frac{\sin(-2)}{-4} = \frac{\sin 2}{4}.$$

Therefore

$$\int_C \frac{\sin z}{z^2 - 4} dz = \pi i \sin 2.$$

(b) Let $f(z) = \frac{1}{z^2 \sin z}$. The isolated singularities of $f$ are $n\pi$ with $n \in \mathbb{Z}$. But only 0 is inside $C$. So,

$$\int_C \frac{1}{z^2 \sin z} dz = 2\pi i \text{Res}(f, 0).$$

To compute $\text{Res}(f, 0)$, note that

$$z^2 \sin z = z^2\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right) = z^3 - \frac{1}{6}z^5 + \frac{1}{120}z^7 - \cdots .$$

By long division,

$$\frac{1}{z^2 \sin\, z} = z^{-3} + \frac{1}{6}z^{-1} \pm \cdots .$$

Therefore

$$\text{Res}(f,0) = \frac{1}{6}.$$

So,

$$\int_C \frac{1}{z^2 \sin\, z} dz = 2\pi i \text{Res}(f,0) = \frac{\pi i}{3}.$$

(c) Let $f(z) = e^{1/z} \sin\left(\frac{1}{z}\right)$. Then the only isolated singularity of $f$ is 0. Now,

$$\begin{aligned} & e^{1/z} \sin\left(\frac{1}{z}\right) \\ = & \left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots\right)\left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots\right) \end{aligned}$$

The coefficient of $\frac{1}{z}$ is 1 Therefore

$$\int_C e^{1/z} \sin\left(\frac{1}{z}\right) dz = 2\pi i \text{Res}(f,0) = 2\pi i.$$