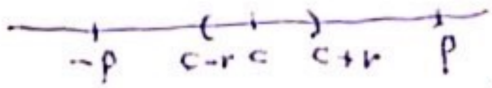
Singular Integrals on $(-\infty,\infty)$

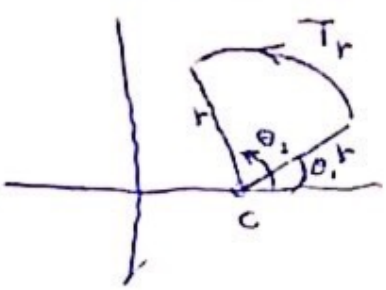
Let $f:(-\infty,\infty)\to\mathbb{C}$ be a continuous function except at $c_1, c_2, \dots, c_N$. For simplicity, assume that f has a singularity at c only.



Then

$$pv\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{\rho\to\infty \\ r\to 0+}} \left\{ \int_{-\rho}^{c-r} f(x)\,dx + \int_{c+r}^{\rho} f(x)\,dx \right\}.$$

Lemma Let $w=f(z)$ be a complex-valued function with a simple pole at c.



$T_r : z = c + re^{i\theta},\ \theta_1 \le \theta \le \theta_2.$

Then

$$\lim_{r\to 0}\int_{T_r} f(z)\,dz = i(\theta_2-\theta_1)\,\mathrm{Res}(f,c).$$

Proof The Laurent series is

$$f(z) = \frac{a_{-1}}{z-c} + \underbrace{\sum_{n=0}^{\infty} a_n(z-c)^n}_{g(z)},\quad 0<|z-c|<R.$$

So, for $r<R$,

$$\int_{T_r} f(z)\,dz = a_{-1}\int_{T_r}\frac{1}{z-c}\,dz + \int_{T_r} g(z)\,dz.$$

Now,

$$a_{-1}\int_{T_r}\frac{1}{z-c}\,dz = a_{-1}\int_{\theta_1}^{\theta_2}\frac{1}{\cancel{re^{i\theta}}}\,i\cancel{re^{i\theta}}\,d\theta$$
$$= i(\theta_2-\theta_1)\,\mathrm{Res}(f,c).$$

Next, suppose $|g(z)| \leq M$ for all z with $|z-c| < R_1$, $R_1 < R$. Then

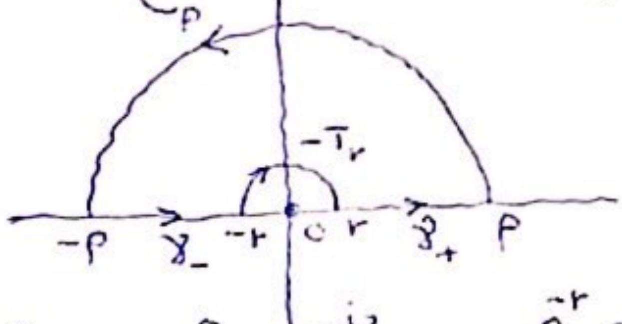
$$\int_{T_r} |g(z)|\,dz \leq M(\theta_2-\theta_1)\,r \to 0$$

as $r \to 0$. $\therefore$

$$\lim_{r\to 0}\int_{T_r} f(z)\,dz = i(\theta_2-\theta_1)\,\mathrm{Res}(f,c).$$

<u>Example</u> Compute $I = \mathrm{pv}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$.

<u>Solution</u>



$\Gamma_{\rho,r} = C_\rho^+ + \gamma_- - T_r + \gamma_+$

Cauchy's integral theorem

$$0 = \int_{\Gamma_{\rho,r}} \frac{e^{iz}}{z}\,dz = \int_{C_\rho^+}\frac{e^{iz}}{z}\,dz + \int_{-\rho}^{-r}\frac{e^{iz}}{z}\,dz - \int_{T_r}\frac{e^{-iz}}{z}\,dz + \int_r^\rho \frac{e^{iz}}{z}\,dz$$

Let $\rho \to \infty$, $r \to 0+$. Then

$$0 = \lim_{\substack{\rho\to\infty\\ r\to 0+}}\left\{\int_{-\rho}^{-r}\frac{e^{ix}}{x}\,dx + \int_r^\rho \frac{e^{ix}}{x}\,dx\right\} - \lim_{r\to 0+}\int_{T_r}\frac{e^{iz}}{z}\,dz$$

$$\therefore \mathrm{pv}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dz = i\pi \lim_{z\to 0} e^{iz} = i\pi$$

$$\therefore \mathrm{pv}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \pi.$$