Weak Solutions

24.1[illegible]

Definition: Let $\sigma \in S^m$, $m > 0$. Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Consider $T_\sigma u = f$ in $\mathbb{R}^n$. Let $u \in L^p(\mathbb{R}^n)$ be such that

$$(u, T_\sigma^* \varphi) = (f, \varphi), \quad \varphi \in \mathcal{S}.$$

Then $u \in L^p(\mathbb{R}^n)$ is a weak solution of $T_\sigma u = f$ in $\mathbb{R}^n$ if

$$(u, T_\sigma^* \varphi) = (f, \varphi), \quad \varphi \in \mathcal{S}$$

Weak Solutions and Maximal Operators

Proposition: Let $\sigma \in S^m$, $m > 0$, and $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then $u \in L^p(\mathbb{R}^n)$ is a weak solution of $T_\sigma u = f$ on $\mathbb{R}^n$
$\Longleftrightarrow u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$.

Question: What are the functions f in $L^p(\mathbb{R}^n)$ for which $T_\sigma u = f$ on $\mathbb{R}^n$ has a weak solution u in $L^p(\mathbb{R}^n)$, where $\sigma \in S^m$, $m > 0$?

Theorem: Let $\sigma \in S^m$, $m > 0$. Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then $T_\sigma u = f$ on $\mathbb{R}^n$ has a weak solution $u \in L^p(\mathbb{R}^n) \Longleftrightarrow \exists C > 0 \ni$

$$|(f, \varphi)| \leq C \| T_\sigma^* \varphi \|_{p'}, \quad \varphi \in \mathcal{S}.$$

Proof: Suppose that $T_\sigma u = f$ on $\mathbb{R}^n$ has a weak solution u in $L^p(\mathbb{R}^n)$. Then

$$(u, T_\sigma^* \varphi) = (f, \varphi), \quad \varphi \in \mathcal{S}.$$

By Hölder's inequality,

$$|(f,\varphi)| \le \underbrace{\|u\|_p}_{C} \|T_\sigma^* \varphi\|_{p'}, \quad \varphi \in \mathcal{S}.$$

Conversely, suppose that

$$|(f,\varphi)| \le C \|T_\sigma^* \varphi\|_{p'}, \quad \varphi \in \mathcal{S}.$$

Let $W \subseteq L^{p'}(\mathbb{R}^n)$ be given by
subspace

$$W = \{ T_\sigma^* \varphi : \varphi \in \mathcal{S} \}.$$

We define $F : W \to \mathbb{C}$ by

$$Fw = (\varphi, f), \quad w \in W$$

where φ is any function with $T_\sigma^* \varphi = w$.

Well-defined? Let $w \in W$. Let φ_1, φ_2 be such that

$$\begin{cases} T_\sigma^* \varphi_1 = w, \\ T_\sigma^* \varphi_2 = w. \end{cases}$$

Then $|(\varphi_1 - \varphi_2, f)| \le C \|T_\sigma^* \varphi_1 - T_\sigma^* \varphi_2\|_{p'} = 0$

$$\Rightarrow (\varphi_1, f) = (\varphi_2, f).$$

Next, $\forall w \in W$, then

$$|Fw| = |(\varphi, f)| \le C \|T_\sigma^* \varphi\|_{p'} = C\|w\|_{p'}.$$

By Hahn-Banach theorem, F can be extended to a bounded linear functional

$$F : L^{p'}(\mathbb{R}^n) \to \mathbb{C}.$$

By the Riesz representation theorem, $\exists\, u \in L^p(\mathbb{R}^n)$

$$\Rightarrow Fw = (w, u) = (\varphi, f), \quad \varphi \in \mathcal{S}.$$

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where $\varphi \in \mathcal{S}$ is $\ni$

$$\overline{T}_\sigma^* \varphi = \omega.$$

$\therefore$ $(\overline{T}_\sigma^* \varphi, u) = (\varphi, f)$, $\varphi \in \mathcal{S}$.

$\therefore$ $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} u = f$, i.e.,

$$\overline{T}_\sigma u = f.$$

Or u is a weak solution of $T_\sigma u = f$ on $\mathbb{R}^n$.