Solutions to Assignment 4

12.4. (i) For $s \geq 0$, we can use Plancherel's theorem to get

$$\begin{aligned} H^{s,2} &= \{u \in L^2(\mathbb{R}^n) : J_{-s}u \in L^2(\mathbb{R}^n)\} \\ &= \{u \in L^2(\mathbb{R}^n) : \mathcal{F}^{-1}\sigma_{-s}\mathcal{F}u \in L^2(\mathbb{R}^n)\} \\ &= \{u \in L^2(\mathbb{R}^n) : \sigma_{-s}\hat{u} \in L^2(\mathbb{R}^n)\} \\ &= \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty\right\}. \end{aligned}$$

For (ii), we again use Plancherel's theorem to obtain for u in $H^{s,2}$,

$$\begin{aligned} \|u\|_{s,2} &= \|J_{-s}u\|_2 = \|\mathcal{F}^{-1}\sigma_{-s}\mathcal{F}u\|_2 \\ &= \|\sigma_{-s}\hat{u}\|_2 = \left\{\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right\}^{\frac{1}{2}}. \end{aligned}$$

12.8. Since T_σ is elliptic, we can find a symbol $\tau \in S^{-m}$ such that

$$T_\tau T_\sigma = I + R,$$

where R is infinitely smoothing. So,

$$u = T_\tau T_\sigma u - Ru = T_\tau f - Ru.$$

Since $T_\tau f \in H^{m,p}$ and $Ru \in \bigcap_{s\in\mathbb{R}} H^{s,p}$, we are done.

12.9. Since T_σ is elliptic, we can find a symbol τ in S^{-m} such that

$$T_\sigma T_\tau = I + S,$$

where S is infinitely smoothing. Let $u = T_\tau f$. Then $u \in H^{m,p}$ and

$$T_\sigma u - f = T_\sigma T_\tau f - f = Sf \in \bigcap_{s\in\mathbb{R}} H^{s,p}.$$

12.10. Since $\hat{\delta} = (2\pi)^{-n/2}$, it follows that

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{\delta}(\xi)|^2 d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|^2)^s d\xi < \infty$$

if and only if $s < -\frac{n}{2}$. Therefore

$$\delta \in H^{s,2} \Leftrightarrow s < -\frac{n}{2}.$$

12.12. Taking the Fourier transform on both sides of

$$(I-\Delta)^{s/2} u = \delta,$$

we have

$$(1+|\xi|^2)^{s/2} \hat{u}(\xi) = (2\pi)^{-n/2}, \quad \xi \in \mathbb{R}^n.$$

So,

$$\hat{u}(\xi) = (2\pi)^{-n/2} (1+|\xi|^2)^{-s/2}, \quad \xi \in \mathbb{R}^n.$$

Taking the inverse Fourier transform of the preceding equation, we get

$$u = (2\pi)^{-n/2} G_s.$$

12.13. While it is true that $\mathcal{S} \subseteq \bigcap_{s\in\mathbb{R}} H^{s,2}$, it is not true that $\bigcap_{s\in\mathbb{R}} H^{s,2} \subseteq \mathcal{S}$. First,

$$\bigcap_{s\in\mathbb{R}} H^{s,2} = \bigcap_{k\in\mathbb{N}} H^{k,2}.$$

Indeed, it is obvious that

$$\bigcap_{s\in\mathbb{R}} H^{s,2} \subseteq \bigcap_{k\in\mathbb{N}} H^{k,2}.$$

Now, let $u \in \bigcap_{k\in\mathbb{N}} H^{k,2}$. Let $s \in \mathbb{R}$. Then let $k \in \mathbb{N}$ be such that $k > s$. By the Sobolev embedding theorem, $H^{k,2} \subseteq H^{s,2}$. But $u \in H^{k,2}$, so $u \in H^{s,2}$.

Since s is an arbitrary real number, it follows that $u \in \bigcap_{s\in\mathbb{R}} H^{s,2}$. Let u be the function on $\mathbb{R}^n$ defined by

$$u(x) = (1+|x|^2)^{-n/2}, \quad x \in \mathbb{R}^n.$$

Let $k \in \mathbb{N}$. Then for all multi-indices α with $|\alpha| = k$, we can find a positive number C_α such that

$$|(D^\alpha u)(x)| \le C_\alpha (1+|x|^2)^{(-n-|\alpha|)/2}, \quad x \in \mathbb{R}^n.$$

Hence

$$\begin{aligned}
& \int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{u}(\xi)|^2 d\xi \\
\le \; & 2^k \int_{\mathbb{R}^n} (1+|\xi|^{2k}) |\hat{u}(\xi)|^2 d\xi \\
= \; & 2^k \|\hat{u}\|_2^2 + 2^k \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{u}(\xi)|^2 d\xi \\
= \; & 2^k \|u\|_2^2 + 2^k \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{u}(\xi)|^2 d\xi \\
= \; & 2^k \|u\|_2^2 + 2^k \int_{\mathbb{R}^n} n^k \sum_{|\gamma|=k} |\xi^\gamma \hat{u}(\xi)|^2 d\xi \\
= \; & 2^k \|u\|_2^2 + (2n)^k \sum_{|\gamma|=k} \int_{\mathbb{R}^n} |(D^\gamma u)^\wedge(\xi)|^2 d\xi \\
= \; & 2^k \|u\|_2^2 + (2n)^k \sum_{|\gamma|=k} \int_{\mathbb{R}^n} |(D^\gamma u)(x)|^2 dx \\
= \; & 2^k \|u\|_2^2 + (2n)^k \sum_{|\gamma|=k} \|D^\gamma u\|_2^2 < \infty.
\end{aligned}$$

Therefore $u \in H^{k,2}$. Since k is an arbitrary positive integer, it follow that

$$u \in \bigcap_{k\in\mathbb{N}} H^{k,2} = \bigcap_{s\in\mathbb{R}} H^{s,2}.$$

Finally,

$$\sup_{x\in\mathbb{R}^n} [(1+|x|^2)^n |u(x)|] = \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{n/2} = \infty.$$

So, $u \notin \mathcal{S}$. This proves that $\bigcap_{s\in\mathbb{R}} H^{s,2}$ is not contained in $\mathcal{S}$.

15.1. By Lemma 15.3, $P(x,D)$ is elliptic. So, there exists a symbol $\tau \in S^{-m}$ such that
$$T_\tau P(x,D) = I + R,$$
where R is infinitely smoothing. Let $f \in H^{s,p}$ and let $u \in \cup_{t\in\mathbb{R}} H^{t,p}$ be any solution of $P(x,D)u = f$ on $\mathbb{R}^n$. Then
$$u + Ru = T_\tau P(x,D)u = T_\tau f.$$
So,
$$u = T_\tau f - Ru.$$
Since $f \in H^{s,p}$ and $\tau \in S^{-m}$, we have
$$T_\tau f \in H^{s-(-m),p} = H^{s+m,p}.$$
Let $t \in \mathbb{R}$ be such that $u \in H^{t,p}$. Suppose that R has symbol in $S^{t-s-m,p}$. Then
$$Ru \in H^{t-(t-s-m),p} = H^{s+m,p}.$$
Therefore $u = T_\tau f - Ru \in H^{s+m,p}$.

16.1. let u and f be in $L^p(\mathbb{R}^n)$. Then u is a solution of $T_\sigma u = f$ on $\mathbb{R}^n$ in the distribution sense $\Leftrightarrow$ for all $\varphi \in \mathcal{S}$,
$$\begin{aligned}(u, T_\sigma^*\overline{\varphi}) &= u(\overline{T_\sigma^*\overline{\varphi}}) \\ &= \int_{\mathbb{R}^n} u(x)\overline{(T_\sigma^*\overline{\varphi})(x)}\,dx \\ &= (u, T_\sigma^*\overline{\varphi}) \\ &= (f, \overline{\varphi})\end{aligned}$$
$\Leftrightarrow$ u is a weak solution in $L^p(\mathbb{R}^n)$ of $T_\sigma u = f$ on $\mathbb{R}^n$.

16.2. Let u be a weak solution in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, of $T_\sigma u = f$ on $\mathbb{R}^n$, where $f \in L^p(\mathbb{R}^n)$. Then
$$(u, T_\sigma^*\varphi) = (f, \varphi), \quad \varphi \in \mathcal{S}.$$

Therefore $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1} = f$. Since σ is elliptic, we have $T_{\sigma,0} = T_{\sigma,1}$. So,

$$u \in \mathcal{D}(T_{\sigma,0}) = H^{m,p}.$$

16.3. For all $\varphi \in \mathcal{S}$,

$$\begin{aligned}
((J_{-s}+q)\varphi,\varphi) &= (J_{-s}\varphi,\varphi) + (q\varphi,\varphi) \\
&\geq (J_{-s}\varphi,\varphi) \\
&= (\widehat{J_{-s}\varphi},\hat{\varphi}) \\
&= (\sigma_{-s}\varphi,\varphi) \\
&= ((1+|\cdot|^2)^{s/2}\varphi,\varphi) \\
&\geq (\varphi,\varphi) \\
&= \|\varphi\|_2^2.
\end{aligned}$$

Therefore for all $\varphi \in \mathcal{S}$,

$$\|\varphi\|_2^2 \leq ((J_{-s}+q)\varphi,\varphi) = (\varphi,(J_{-s}+q)^*\varphi) \leq \|\varphi\|_2\|(J_{-s}+q)^*\varphi\|_2,$$

which is the same as

$$\|\varphi\|_2 \leq \|(J_{-s}+q)^*\varphi\|_2.$$

So, for all $\varphi \in \mathcal{S}$,

$$|(f,\varphi)| \leq \|f\|_2\|\varphi\|_2 \leq \|f\|_2\|(J_{-s}+q)^*\varphi\|_2.$$

Thus, $(J_{-s}+q)u = f$ has a weak solution in $L^2(\mathbb{R}^n)$ for all $f \in L^2(\mathbb{R}^n)$.