Let $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ be such that

$$\sup_{x\in\mathbb{R}^n} |(D^\beta a_\alpha)(x)| < \infty$$

for all α with $|\alpha|\le m$ and all β. Then $P(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, $x, \xi \in \mathbb{R}^n$, is in S^m.

<u>Theorem</u>: Suppose $\exists\, x_0 \in \mathbb{R}^n \ni$ we can find $C_1 > 0$ and $C_2 > 0$ for which

$$\begin{cases} \left| \sum_{|\alpha|=m} a_\alpha(x_0)\xi^\alpha \right| \ge C_1 |\xi|^m, \\ \left| \sum_{|\alpha|=m} (a_\alpha(x) - a_\alpha(x_0))\xi^\alpha \right| \le C_2 |\xi|^m \end{cases}$$

for all $x, \xi \in \mathbb{R}^n$ and $C_1 > C_2$. Then for $-\infty < s < \infty$, $1 < p < \infty$,

$$\begin{cases} u \in H^{s,p}, \\ P(x,D)u = f, \\ f \in H^{s,p} \end{cases}$$

$$\Rightarrow u \in H^{s+m,p}.$$

<u>Remarks</u>: 1) Meaning of Theorem: Consider the PDE

$$P(x,D)u = f$$

on $\mathbb{R}^n$, where f is the given data in $H^{s,p}$. Then every solution u in $H^{s,p}$ has to be in $H^{s+m,p}$.

2) Recall that

$$H^{s+m,p} \subset H^{s+m-1,p} \subset \cdots \subset H^{s,p}.$$

Thus, the solution u is m steps more selective or regular than the input f defined globally on $\mathbb{R}^n$. So, we call the theorem a <u>global regularity theorem</u>.

3) The theorem is a nontrivial extension of a result of Hess and Kato.

<u>Lemma</u>: Under the hypotheses of the theorem, $\exists\, C > 0$ and $R > 0 \ni$

$$|P(x,\xi)| \geq C(1+|\xi|)^m, \quad |\xi| \geq R.$$

<u>Remark</u>: The partial differential operator in the theorem is elliptic.

<u>Proof of Lemma</u>: We have

$$\left|\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\right| = \left|\sum_{|\alpha|=m}(a_\alpha(x)-a_\alpha(x_0))\xi^\alpha + \sum_{|\alpha|=m} a_\alpha(x_0)\xi^\alpha\right|$$

$$\geq \left|\sum_{|\alpha|=m} a_\alpha(x_0)\xi^\alpha\right| - \left|\sum_{|\alpha|=m}(a_\alpha(x)-a_\alpha(x_0))\xi^\alpha\right|$$

$$\geq (C_1 - C_2)|\xi|^m, \quad x, \xi \in \mathbb{R}^n.$$

So, $\exists\, C' > 0, C'' > 0$ and $R > 0 \ni$

$$\left|\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha\right| = \left|\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha + \sum_{|\alpha|<m} a_\alpha(x)\xi^\alpha\right|$$

$$\geq (C_1 - C_2)|\xi|^m - \left|\sum_{|\alpha|<m} a_\alpha(x)\xi^\alpha\right|$$

$$\geq C'(1+|\xi|)^m - C''(1+|\xi|)^{m-1}$$

$$= (1+|\xi|)^m\left(C' - C''(1+|\xi|)^{-1}\right), \quad |\xi| \geq R.$$

$\therefore \exists\, R_1 > 0$ with $R_1 > R$ and

$$C' - C''(1+|\xi|)^{-1} \geq \frac{C'}{2}, \quad |\xi| \geq R_1.$$

Therefore

$$\left|\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha\right| \geq \frac{C'}{2}(1+|\xi|)^m, \quad |\xi| \geq R_1.$$

Hence $P(x,D)$ is elliptic.

Proof of the Global Regularity Theorem

Since $P(x,D)$ is elliptic of order m, $\exists\, \tau \in S^{-m} \ni$

$$T_\tau P(x,D) = I + R,$$

where R is infinitely smoothing. So

$$u = T_\tau P(x,D)u - Ru$$
$$= T_\tau f - Ru.$$

$T_\tau : H^{s,p} \xrightarrow[\text{linear}]{\text{bounded}} H^{s-(-m)} \Rightarrow T_\tau f \in H^{s+m,p}$

R is also of order $-m \Rightarrow Ru \in H^{s+m,p}$.

$$\therefore u \in H^{s+m,p}.$$