Trigonometric Integrals

Compute $I = \int_0^{2\pi} U(\cos\theta, \sin\theta)\,d\theta$, where U is a rational function of $\cos\theta$ and $\sin\theta$. To compute I, let C be the unit circle with center 0 and oriented once in the ~~[illegible]~~ counterclockwise direction. Parametrize C as $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Then

$$\begin{cases} \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right), \\ \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{1}{2i}\left(z-\dfrac{1}{z}\right), \\ dz = ie^{i\theta}d\theta \Rightarrow d\theta = \dfrac{1}{iz}dz. \end{cases}$$

So $I = \int_C U\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{1}{iz}dz$, a contour integral.

Example (Exercise 3 in Chapter 14)

Let $a \in \mathbb{C}$ with $|a|<1$. Compute

$$I = \int_0^{2\pi} \frac{1}{1-2a\cos\theta + a^2}\,d\theta.$$

Solution:

$$I = \int_C \frac{1}{1-\cancel{2}a\left(z+\frac{1}{z}\right)\frac{1}{\cancel{2}}+a^2}\frac{1}{iz}dz$$

$$= \frac{1}{i}\int_C \frac{1}{z-a(z^2+1)+a^2z}dz$$

$$= -\frac{1}{i}\int_C \frac{1}{a(z^2+1)-(1+a^2)z}dz = -\frac{1}{i}\int_C \frac{1}{az^2-(1+a^2)z+a}dz$$

Factorizing the denominator

$$z = \frac{(1+a^2)\pm\sqrt{(1+a^2)^2-4a^2}}{2a} = \frac{(1+a^2)\pm\sqrt{1+2a^2+a^4-4a^2}}{2a}$$

$$= \frac{(1+a^2)\pm\sqrt{1-2a^2+a^4}}{2a} = \frac{(1+a^2)\pm\sqrt{(1-a^2)^2}}{2a}$$

$\therefore z = \dfrac{1+a^2 \pm (1-a^2)}{2a} = \dfrac{1}{a}$ or a. We take a only because a is inside C. So,

$$I = -\frac{1}{i}\int_C \frac{1}{a(z-a)(z-\frac{1}{a})}\,dz$$

Let $f(z) = \dfrac{1}{a(z-a)(z-\frac{1}{a})}$. Then a is a simple pole of f.

By Cauchy's Residue Theorem,

$$\begin{aligned} I &= -\frac{1}{i}\,2\pi i\,\mathrm{Res}(f,a) \\ &= -\frac{1}{i}\,2\pi i \lim_{z\to a}(z-a)f(z) = -2\pi \lim_{z\to a}\frac{1}{(z-\frac{1}{a})a} \\ &= -2\pi\frac{a}{(a^2-1)a} = 2\pi\,\frac{\cancel{a}\,1}{1-a^2} \end{aligned}$$

$\therefore I = \dfrac{2\pi}{1-a^2}$.

Cauchy's Principal Values of Improper Integrals on $(-\infty,\infty)$

Let f be a continuous complex-valued function on $[0,\infty)$.

Definition: The Cauchy Principal Value $\mathrm{pv}\int_{-\infty}^{\infty} f(x)\,dx$ of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined by

$$\mathrm{pv}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx,$$

if the limit exists.

Fourier Transforms

Let f be a continuous complex-valued function on $(-\infty,\infty)$. Then we define the Fourier transform $\hat{f}$ of f on $(-\infty,\infty)$ by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\,\mathrm{pv}\int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx, \quad \xi\in(-\infty,\infty)$$

if the Cauchy principal value exists.

Sometimes we omit pv with no confusion.

Closely related to the Fourier transform are the cosine transform and sine transform, respectively given by

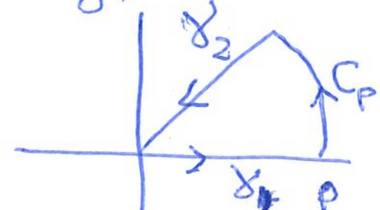
$$\frac{1}{\sqrt{2\pi}}\, pv\int_{-\infty}^{\infty} \cos(x\xi)\, f(x)\,dx \quad \text{and} \quad \frac{1}{\sqrt{2\pi}}\, pv\int_{-\infty}^{\infty} \sin(x\xi)\, f(x)\,dx.$$

Example: Compute $\int_0^{\infty} \cos(x^2)\,dx$ and $\int_{-\infty}^{\infty} \sin(x^2)\,dx$.

(Fresnel Integrals)

(Ex 4 in Chapter 16)

Solution We first compute $\int_C e^{iz^2}\,dz$, where C is the contour given by



$$C = \gamma_1 + C_\rho + \gamma_2,$$

so by Cauchy's integral theorem,

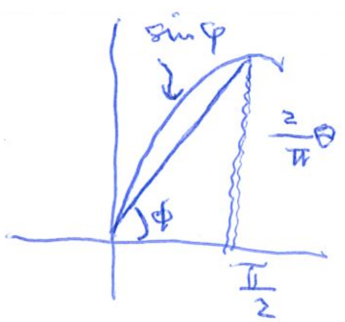
$$0 = \int_C e^{iz^2}\,dz = \int_0^{\rho} e^{ix^2}\,dx + \int_0^{\pi/4} e^{i\rho^2 e^{2i\theta}}\, i\rho e^{i\theta}\,d\theta - e^{i\frac{\pi}{4}} \int_0^{\rho} e^{-r^2}\,dr.$$

Now,

$$\left| \int_0^{\pi/4} e^{i\rho^2 e^{2i\theta}}\, i\rho e^{i\theta}\,d\theta \right| \le \rho \int_0^{\pi/4} e^{-\rho^2 \sin 2\theta}\,d\theta$$

$$\le \rho \int_0^{\frac{\pi}{4}} e^{-\rho^2 4\theta/\pi}\,d\theta$$

$$= \rho \left. \frac{-e^{-\frac{\rho^2 4}{\pi}\theta}}{\frac{\rho^2 4}{\pi}} \right|_0^{\frac{\pi}{4}}$$



$$= \rho\left(1 - e^{-\rho^2}\right) \Big/ \left(\frac{\rho^2 4}{\pi}\right)$$

$$= \frac{\pi}{4\rho}\left(1 - e^{-\rho^2}\right) \to 0$$

$$\therefore \int_0^{\infty} e^{ix^2}\,dx = e^{i\frac{\pi}{4}} \int_0^{\infty} e^{-x^2}\,dx \quad \text{as } \rho \to \infty.$$

$$\int_0^\infty e^{ix^2}\,dx = e^{i\pi/4}\sqrt{\pi}$$

$$= \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)\sqrt{\pi}$$

$$\int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2}}{2}\sqrt{\pi} = \sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty \sin(x^2)\,dx = \sqrt{\frac{\pi}{2}}$$