The Maximal Operator

Definition: Let $u, f \in L^p(\mathbb{R}^n)$, $1<p<\infty$. We say that $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}u = f \Leftrightarrow (u, T_\sigma^* \varphi) = (f, \varphi)$, $\varphi \in \mathcal{S}$.

Proposition: Let $u \in \mathcal{D}(T_{\sigma,1})$. Then $T_{\sigma,1}u = T_\sigma u$ in $\mathcal{S}'$.

Proof: Since

$$(T_{\sigma,1}u, \varphi) = (u, T_\sigma^* \varphi), \quad \varphi \in \mathcal{S},$$

we have

$$\begin{aligned}(T_{\sigma,1}u)(\bar{\varphi}) &= u(\overline{T_\sigma^* \varphi}), \quad \varphi \in \mathcal{S},\\ &= u(\overline{T_\sigma^* \bar{\bar{\varphi}}}), \quad \varphi \in \mathcal{S},\\ &= (T_\sigma u)(\bar{\varphi}), \quad \varphi \in \mathcal{S}.\end{aligned}$$

$\therefore T_{\sigma,1}u = T_\sigma u$ in $\mathcal{S}'$.

Proposition: $T_{\sigma,1} : \mathcal{D}(T_{\sigma,1}) \longrightarrow L^p(\mathbb{R}^n)$ is a closed linear operator with $\mathcal{S} \subset \mathcal{D}(T_\sigma)$.

$$\mathcal{D}(T_{\sigma,1}) \underset{\text{dense subspace}}{\cap} L^p(\mathbb{R}^n)$$

Proof: Let $u \in \mathcal{S}$.

Then $(u, T_\sigma^* \varphi) = (T_\sigma u, \varphi)$, $\varphi \in \mathcal{S}$. $\therefore u \in \mathcal{D}(T_{\sigma,1})$, $T_{\sigma,1}u = T_\sigma u$.

Let $\{u_k\}_{k=1}^\infty$ be a sequence in $\mathcal{D}(T_{\sigma,1}) \ni u_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_{\sigma,1}u_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Then $\forall \varphi \in \mathcal{S}$,

$$\begin{array}{ccc} (u_k, T_\sigma^* \varphi) & = & (T_{\sigma,1}u_k, \varphi) \\ \downarrow & & \downarrow \\ (u, T_\sigma^* \varphi) & = & (f, \varphi). \end{array}$$

$\therefore u \in \mathcal{D}(T_{\sigma,1})$, $T_{\sigma,1}u = f$.

Proposition: $\mathcal{S} \subseteq \mathcal{D}(T_{\sigma,1}^t)$.

Proof: $T_{\sigma,1}^t : \mathcal{D}(T_{\sigma,1}^t) \longrightarrow L^{p'}(\mathbb{R}^n)$ is $\ni \forall \psi \in \mathcal{S}$,

$$\mathcal{D}(T_{\sigma,1}^t) \cap L^{p'}(\mathbb{R}^n)$$

$$(\psi, T_{\sigma,1}u) = (T_\sigma^* \psi, u), \quad u \in \mathcal{D}(T_{\sigma,1})$$

$\therefore \psi \in \mathcal{D}(T_{\sigma,1}^t)$,

$T_{\sigma,1}^t \psi = T_\sigma^* \psi$.

<u>Proposition</u>: $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$.

<u>Proof</u>: Let $u \in \mathcal{D}(T_{\sigma,0})$ and $T_{\sigma,0}u = f$. Then $\exists$ a sequence $\{\varphi_k\}_{k=1}^{\infty}$ in $\mathcal{S}$ $\ni$ $\varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_\sigma \varphi_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. So, $\forall \psi \in \mathcal{S}$,

$$(u, T_\sigma^* \psi) = \lim_{k\to\infty} (\varphi_k, T_\sigma^* \psi) = \lim_{k\to\infty} (T_\sigma \varphi_k, \psi)$$
$$= (f, \psi).$$

$\therefore$ $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}u = f$.

<u>Remark</u>: $T^t_{\sigma,0}$ is an extension of $T^t_{\sigma,1}$. Since $\mathcal{S} \subseteq \mathcal{D}(T^t_{\sigma,1})$, we have

$$\mathcal{S} \subseteq \mathcal{D}(T^t_{\sigma,0}).$$

<u>Proposition</u> ⊛: $T_{\sigma,1}$ is the largest closed extension of T_σ in the sense that if B is any extension of T_σ with $\mathcal{S} \subseteq \mathcal{D}(B^t)$, then $T_{\sigma,1}$ is an extension of B.

<u>Lemma</u>: $\forall \varphi \in \mathcal{S}$, $T_\sigma^t \varphi = T_\sigma^* \varphi$.

<u>Proof</u>: Let $\varphi \in \mathcal{S}$. Then

$$(T_\sigma^* \varphi, \psi) = (\varphi, T_\sigma \psi), \quad \psi \in \mathcal{S}$$
$$\parallel$$
$$(T_\sigma^t \varphi, \psi)$$

$\therefore$ $T_\sigma^t = T_\sigma^*$ on $\mathcal{S}$.

<u>Proof of Proposition ⊛</u>: Let $u \in \mathcal{D}(B)$. Then $\forall \psi \in \mathcal{S}$, $\psi \in \mathcal{D}(B^t)$. $\therefore$

$$(\psi, Bu) = (B^t \psi, u) = (T_\sigma^t \psi, u) = (T_\sigma^* \psi, u)$$

$\therefore$ $(u, T_\sigma^* \psi) = (Bu, \psi)$, $\psi \in \mathcal{S}$.

$\therefore$ $u \in \mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}u = Bu$.

Remarks: We call $T_{\sigma,1}$ the maximal operator of T_σ. We know that $T_{\sigma,1}$ is an extension of $T_{\sigma,0}$.

Question: Is $T_{\sigma,0} = T_{\sigma,1}$?

Theorem A: Let $\sigma \in S^m$, $m > 0$, be elliptic. Then $\mathcal{D}(T_{\sigma,0}) = H^{m,p}$.

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Theorem B: $\exists\, C_1, C_2 > 0 \ni$

$$C_1 \|u\|_{m,p} \le \left(\|T_\sigma u\|_{0,p} + \|u\|_{0,p} \right) \le C_2 \|u\|_{m,p}, \quad u \in H^{m,p}$$

(Agmon-Douglis-Nirenberg)

Proof: The second inequality is the L^p-boundedness and the Sobolev Embedding Theorem. Since σ is elliptic, $\exists\, \tau \in S^{-m}$ $\ni$

$$T_\tau T_\sigma = I + R, \text{ where } R \text{ is infinitely smoothing.}$$

$\therefore$

$$u = T_\tau T_\sigma u - Ru, \quad u \in H^{m,p}$$

$\therefore$

$$\|u\|_{m,p} \le \|T_\tau T_\sigma u\|_{m,p} + \|Ru\|_{m,p}$$
$$\le C\left(\|T_\sigma u\|_{0,p} + \|u\|_{0,p} \right)$$

Theorem C: $\mathcal{S}$ is dense in $H^{s,p}$, $-\infty < s < \infty$, $1 < p < \infty$.

Proof: Let $u \in H^{s,p}$. Then $J_{-s} u \in L^p(\mathbb{R}^n)$. $\exists\, \varphi_k \in \mathcal{S} \ni \varphi_k \to J_{-s} u$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Let

$$\psi_k = J_s \varphi_k, \quad k \in \mathbb{N}.$$

Then $\psi_k \in \mathcal{S}$, and

$$\|\psi_k - u\|_{s,p} = \|J_{-s}\psi_k - J_{-s} u\|_p = \|\varphi_k - J_{-s} u\|_p \to 0 \text{ as } k \to \infty.$$

$\therefore$ $\mathcal{S}$ is dense in $H^{s,p}$.

Proof of Theorem A

Let $u \in H^{m,p}$. Then $\exists\, \varphi_k \in \mathcal{S} \ni \varphi_k \to u$ in $H^{m,p}$ as $k \to \infty$. By the ADN inequalities,

$$\|T_\sigma(\varphi_k - \varphi_j)\|_p + \|\varphi_k - \varphi_j\|_p \le C_2 \|\varphi_k - \varphi_j\|_{m,p} \to 0 \text{ as } k, j \to \infty.$$

$\therefore$ $\{T_\sigma \varphi_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ are Cauchy sequences in $L^p(\mathbb{R}^n)$. So,

$$\varphi_k \to u \text{ and } T_\sigma \varphi_k \to f$$

in $L^p(\mathbb{R}^n)$ as $k \to \infty$. $\therefore$ $u \in \mathcal{D}(T_{\sigma,0})$. $\therefore$ $H^{m,p} \subseteq \mathcal{D}(T_{\sigma,0})$. Now, let $u \in \mathcal{D}(T_{\sigma,0})$. Then $\exists\, \varphi_k \in \mathcal{S} \ni \varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_\sigma \varphi_k \to T_{\sigma,0} u$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. $\therefore$

$$\|\varphi_j - \varphi_k\|_{m,p} \le \frac{1}{C_1}\left(\|T_\sigma \varphi_j - T_\sigma \varphi_k\|_p + \|\varphi_j - \varphi_k\|_p\right) \to 0$$

as $j, k \to \infty$. $\therefore$ $\varphi_k \to v$ for some v in $H^{m,p}$ as $k \to \infty$. By Sobolev Embedding Theorem, $\varphi_k \to v$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. $\therefore$ $u = v$. $\therefore$ $u \in H^{m,p}$. $\therefore$ $\mathcal{D}(T_{\sigma,0}) \subseteq H^{m,p}$.

So, $\mathcal{D}(T_{\sigma,0}) = H^{m,p}$.