Answers to Test 2

1 No. 1 Let $f : \mathbb{C} \to \mathbb{C}$ be the function given by

$$f(z) = z, \quad z \in \mathbb{C}.$$

Then f is entire, but

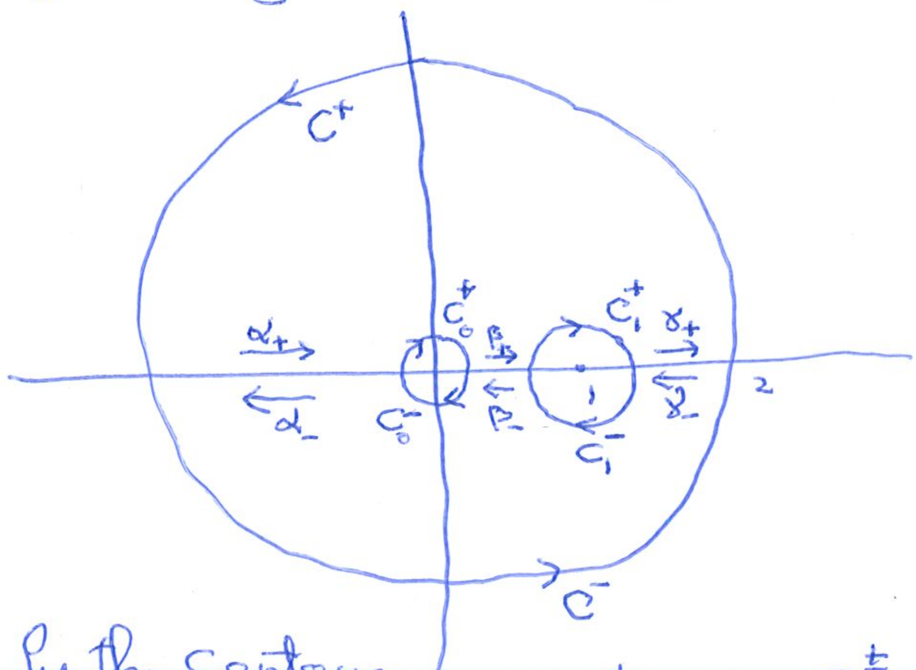
1 $$g(z) = f(\bar{z}) = \bar{z}, \quad z \in \mathbb{C},$$

which is nowhere holomorphic. If g has an antiderivative G, then

2 $$G'(z) = g(z), \quad z \in \mathbb{C},$$

showing that g is entire. This is a contradiction.

2



Let $C_\pm$ be the contour given by

$$C_\pm = C^\pm + \alpha_\pm + C_0^\pm + \beta_\pm + C_1^\pm + \gamma_\pm .$$

Then by Cauchy's integral theorem,

1 $$\left(\int_{C^+} + \int_{-C_0} + \int_{-C_1} + \int_{C^-} \right) \left(\frac{z+1}{z^3 - z^2} \right) dz = 0.$$

$$\therefore \int_C \frac{z+1}{z^3 - z^2}\, dz = \int_{C_0} \frac{z+1}{z^3 - z^2}\, dz + \int_{C_1} \frac{z+1}{z^3 - z^2}\, dz.$$

By Cauchy's Integral Formula,

$$\int_{C_0} \frac{z+1}{z^3-z^2}\,dz = \int_{C_0} \frac{\frac{z+1}{z+1}}{z^2}\,dz$$

$$= -2\pi i \left.\frac{d}{dz}\right|_{z=0} \left(\frac{z+1}{z-1}\right)$$

$$= -2\pi i \left.\frac{(z-1)-(z+1)}{(z-1)^2}\right|_{z=0}$$

$$= 4\pi i. \quad (1)$$

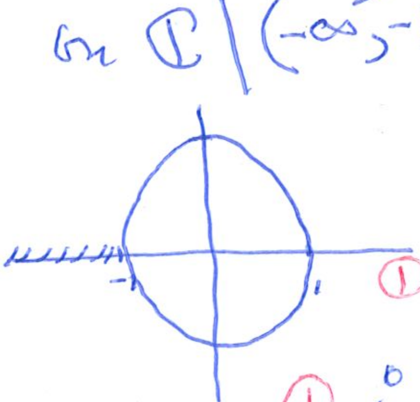
By Cauchy's Integral Formula again,

$$\int_{C_1} \frac{z+1}{z^3-z^2}\,dz = \int_{C_1} \frac{\frac{z+1}{z^2}}{z-1}\,dz = -4\pi i \quad (1)$$

$$\therefore \int_{C} \frac{z+1}{z^3-z^2}\,dz = 0. \quad (1)$$

(3) Let $f(z) = \mathrm{Log}(1+z)$. Then f is holomorphic on $\mathbb{C} \setminus (-\infty, -1]$. (1) $f(0) = 0$ and for all $n \in \mathbb{N}$,

$$f^{(n)}(z) = (-1)^{n+1} \frac{(n-1)!}{(1+z)^n}, \quad |z|<1.$$

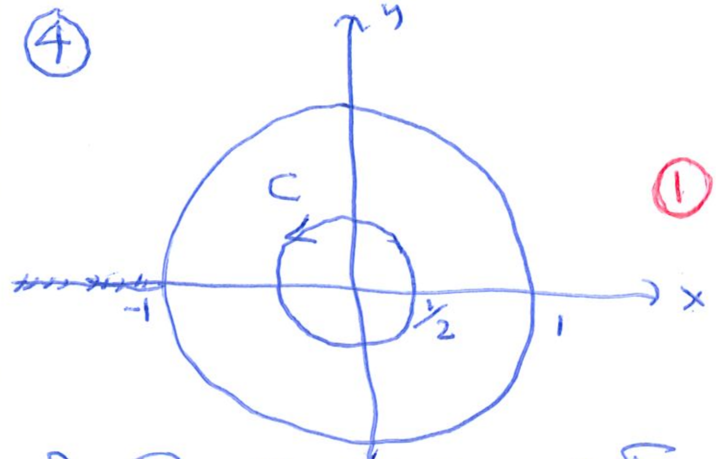


(1) $\therefore f^{(n)}(0) = (-1)^{n+1}(n-1)!$

(1) $\therefore \mathrm{Log}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}.$

The largest disk of convergence for the Taylor series is $\{z \in \mathbb{C} : |z|<1\}$. (1)

(4)



(1)

Log(1+z) is holomorphic on a simply connected domain and C is a simple closed contour lying in the simply connected domain. (3)

By Cauchy's Integral Formula,

(2) $$\int_C \frac{\operatorname{Log}(1+z)}{z^2}\,dz = 2\pi i \left.\frac{d}{dz}\right|_{z=0} \operatorname{Log}(1+z)$$

$$= 2\pi i$$ (1)

(5) Let f be the holomorphic function with power series $\sum_{n=0}^{\infty} a_n z^n$. Then

(1) $$f(z) = \sum_{n=0}^{\infty} 2^{2n} z^{2n} + \sum_{n=0}^{\infty} 3 z^{2n+1}$$

(1) $$= \sum_{n=0}^{\infty} (4z^2)^n + \left(\sum_{n=0}^{\infty} z^{2n}\right) 3z$$

(1) $$= \frac{1}{1-4z^2} + \frac{3z}{1-z^2}, \quad |4z^2| < 1$$
i.e., $|z| < \frac{1}{2}$.

∴ the radius of convergence is $\frac{1}{2}$. (1)