Let $A : \mathcal{D}(A) \to Y$ be a linear operator
$\underset{X}{\overset{\cap|}{}}$ dense subspace

We define $A^t : \mathcal{D}(A^t) \to X'$ to be the mapping
$\underset{Y'}{\overset{\cap|}{}}$

as follows:

- Let Z be a Banach space with norm $\| \ \|_Z$.
  Then $Z' = \{$all bounded conjugate linear functionals on $Z\}$.
  Z' is a Banach space over $\mathbb{C}$ with norm $\| \ \|_{Z'}$ given by
  $$\|f\|_{Z'} = \sup_{\substack{z \in Z \\ z \neq 0}} \frac{|f(z)|}{\|z\|_Z}, \quad f \in Z'.$$
  We call Z' the <u>dual</u> of Z.
- $\mathcal{D}(A^t) = \{ y' \in Y' : \exists\, x' \in X' \ni y'(Ax) = x'(x) \ \forall x \in \mathcal{D}(A) \}$.

  <u>Lemma</u>: $\forall y' \in Y'$, $\exists$ at most one $x' \in X'$ $\ni y'(Ax) = x'(x)$, $\forall x \in \mathcal{D}(A)$.

  <u>Proof</u>: Suppose that x' and z' are in X' $\ni$
  $$\begin{cases} y'(Ax) = x'(x), & x \in \mathcal{D}(A), \\ y'(Ax) = z'(x), & x \in \mathcal{D}(A). \end{cases}$$
  Then $x'(x) = z'(x)$, $x \in \mathcal{D}(A)$.
  ~~$\therefore (x' - z')(x), x \in \mathcal{D}(A)$~~

  $\therefore$ Since $\mathcal{D}(A)$ is dense in X, $x'(x) = z'(x)$, $x \in X$.

  $\therefore x' = z'$.
- $\forall y' \in \mathcal{D}(A^t)$, we define $A^t y'$ to be x', where x' is given in the definition of $\mathcal{D}(A^t)$.

We call A^t the <u>adjoint</u> or <u>true adjoint</u> of A. 20.2

Let $A : \mathcal{D}(A) \to Y$ be a linear operator,
$\mathcal{D}(A) \subseteq X$ dense subspace
(closed or unclosed). Then $A^t : \mathcal{D}(A^t) \to X'$, $\mathcal{D}(A^t) \subseteq Y'$
is a closed linear operator.

<u>Proof</u>: Let $\{y'_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{D}(A^t)$

$\ni \begin{cases} A^t y'_k \to x' \text{ in } X', \\ y'_k \to y' \text{ in } Y'. \end{cases}$

Then
$$y'_k(Ax) = (A^t y'_k)(x), \quad x \in \mathcal{D}(A),$$
$$\downarrow \qquad\qquad \downarrow$$
$$y'(Ax) = x'(x) \quad x \in \mathcal{D}(A).$$

$\therefore y' \in \mathcal{D}(A^t)$ and $A^t y' = x'$. $\therefore A^t$ is closed.

<u>Proposition</u> Let $A : \mathcal{D}(A) \to Y$ be a linear operator, $\mathcal{D}(A) \subseteq X$ dense subspace.
Let $B : \mathcal{D}(B) \to Y$ be an extension of A, $\mathcal{D}(B) \subseteq X$. Then
A^t is an extension of B^t.

<u>Proof</u>: Let $y' \in \mathcal{D}(B^t)$ and $B^t y' = x'$. Then $\forall x \in \mathcal{D}(B)$,
$$y'(Bx) = x'(x).$$
Since B is an extension of A,
$$y'(Ax) = x'(x), \quad x \in \mathcal{D}(A).$$
$\therefore y' \in \mathcal{D}(A^t)$ and $A^t y' = x'$. $\therefore A^t$ is an extension of B^t.

Back to Pseudo-Differential Operators

Let $\sigma \in S^m$, $m > 0$. Then for $-\infty < s < \infty$ and $1 < p < \infty$,

$$T_\sigma : H^{s,p} \to H^{s-m,p}$$

is a bounded linear operator. Let $s = 0$. Then $H^{0,p} = L^p(\mathbb{R}^n)$. So,

$$T_\sigma : \underbrace{H^{0,p}}_{L^p(\mathbb{R}^n)} \to H^{-m,p}$$

Since $L^p(\mathbb{R}^n) \subset H^{-m,p}$, $\therefore T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator. It is not even closed. So, we look at

$$T_\sigma : \begin{array}{l}\mathcal{S}\\ \cap \ \text{dense subspace}\\ L^p(\mathbb{R}^n)\end{array} \longrightarrow L^p(\mathbb{R}^n). \quad \text{(not closed either)}$$

Is it closable?

Proposition: $T_\sigma : \begin{array}{l}\mathcal{S}\\ \cap \ \text{dense subspace}\\ L^p(\mathbb{R}^n)\end{array} \longrightarrow L^p(\mathbb{R}^n)$ is closable.

Proof: Let $\{\varphi_k\}_{k=1}^\infty$ be a sequence in $\mathcal{S}$ such that $\varphi_k \to 0$ in $L^p(\mathbb{R}^n)$ and $T_\sigma \varphi_k \to \psi$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Then $\forall \varphi \in \mathcal{S}$,

$$\begin{array}{ccc}(T_\sigma \varphi_k, \varphi) & = & (\varphi_k, T_\sigma^* \varphi)\\ \downarrow & & \downarrow\\ (\psi, \varphi) & = & (0, T_\sigma^* \varphi)\end{array}$$

$\therefore (\psi, \varphi) = 0$. $\therefore \psi = 0$. $\therefore T_\sigma$ is closable.

Conclusion

The minimal operator of T_σ exists, denoted by $T_{\sigma,0}$.

$$\mathcal{D}(T_{\sigma,0}) = \{ u \in L^p(\mathbb{R}^n) : \exists\, \varphi_k \in \mathcal{S} \ni \varphi_k \to u \text{ in } L^p(\mathbb{R}^n) \text{ and } T_\sigma \varphi_k \to f \text{ for some } f \text{ in } L^p(\mathbb{R}^n) \text{ as } k \to \infty \}.$$

Also,

$\forall u \in \mathcal{D}(T_{\sigma,0})$,

$$T_{\sigma,0} u = f.$$