Solutions to Assignment 3

7.2. For $j = 1, 2, \ldots, n$, we pick $\gamma^{(j)}$ to be the multi-index with N in the j^{th} position and zeros elsewhere. Then

$$\sum_{|\gamma|=N} |z^\gamma|^2 = \sum_{|\gamma|=N} z_1^{2\gamma_1} z_2^{2\gamma_2} \cdots z_n^{2\gamma_n} \geq z_j^{2N}$$

for $j = 1, 2, \ldots, n$. Therefore

$$\left\{ \sum_{|\gamma|=N} |z^\gamma|^2 \right\}^{1/N} \geq z_j^2$$

for $j = 1, 2, \ldots, n$. Summing over j from 1 to n, we get

$$n \left\{ \sum_{|\gamma|=N} |z^\gamma|^2 \right\}^{1/N} \geq \sum_{j=1}^{n} |z_j|^2 = |z|^2$$

and hence

$$n^N \sum_{|\gamma|=N} |z^\gamma|^2 \geq |z|^{2N}.$$

8.4. Let $T_{\lambda_1} = T_\sigma T_\tau$ and $T_{\lambda_2} = T_\tau T_\sigma$. Then

$$T_\sigma T_\tau - T_\tau T_\sigma = T_{\lambda_1 - \lambda_2}.$$

By the product formula with $N = 1$,

$$\lambda_1 - \sigma\tau \in S^{m_1+m_2-1}$$

and

$$\lambda_2 - \tau\sigma \in S^{m_1+m_2-1}.$$

Therefore

$$\lambda_1 - \lambda_2 \in S^{m_1+m_2-1}.$$

9.1. Suppose that a pseudo-differential operator T_σ has two formal adjoints T_1 and T_2. Then for all φ and ψ in $\mathcal{S}$,

$$(T_1\varphi, \psi) = (\varphi, T_\sigma\psi)$$

and

$$(T_2\varphi, \psi) = (\varphi, T_\sigma\psi).$$

So,

$$((T_1 - T_2)\varphi, \psi) = 0.$$

Therefore

$$(T_1 - T_2)\varphi = 0, \quad \varphi \in \mathcal{S},$$

and hence$T_1 - T_2 = 0$, i.e., $T_1 = T_2$.

9.2. For all φ and ψ in $\mathcal{S}$,

$$(((T_\sigma)^*)^*\varphi, \psi) = (\varphi, T_\sigma^*\psi) = (T_\sigma\varphi, \psi).$$

Therefore $(T_\sigma^*)^* = T_\sigma$. Also

$$((T_\sigma T_\tau)^*\varphi, \psi) = (\varphi, T_\sigma T_\tau\psi) = (T_\sigma^*\varphi, T_\tau\psi) = (T_\tau^* T_\sigma^*\varphi, \psi).$$

Therefore $(T_\sigma T_\tau)^* = T_\tau^* T_\sigma^*$.

10.1. (i) Suppose that $P(x, D)$ is elliptic. Then there exist positive constants C and R such that

$$|P(x, \xi)| \geq C(1 + |\xi|)^m, \quad |\xi| \geq R.$$

So, we can find another positive constant C' such that

$$\begin{aligned} |P_m(x, \xi)| &= \left| P(x, \xi) - \sum_{|\alpha| < m} a_\alpha(x)\xi^\alpha \right| \\ &\geq |P(x, \xi)| - \left| \sum_{|\alpha| < m} a_\alpha(x)\xi^\alpha \right| \end{aligned}$$

$$
\begin{aligned}
&\geq |P(x,\xi)| - \sum_{|\alpha|<m} |a_\alpha(x)|(1+|\xi|)^{|\alpha|} \\
&\geq C(1+|\xi|)^m - C'(1+|\xi|)^{m-1} \\
&= C(1+|\xi|)^m \left(1 - \frac{C'}{C}(1+|\xi|)^{-1}\right)
\end{aligned}
$$

whenever $|\xi| \geq R$. Hence there exists another positive constant R' such that

$$|P_m(x,\xi)| \geq \frac{C}{2}(1+|\xi|)^m, \quad |\xi| \geq R'.$$

Conversely, suppose that there exist positive constants C and R such that

$$|P_m(x,\xi)| \geq C(1+|\xi|)^m, \quad |\xi| \geq R.$$

Then

$$|P(x,\xi)| = \left| P_m(x,\xi) + \sum_{|\alpha|<m} a_\alpha(x)\xi^\alpha \right| \geq |P(x,\xi)| - \sum_{|\alpha|<m} |a_\alpha(x)|(1+|\xi|)^{|\alpha|}$$

for all x and ξ in $\mathbb{R}^n$ and the proof is then similar to that of the direct implication. (ii) Suppose that $P(D)$ is elliptic. Then there exist positive constants such that

$$|P_m(\xi)| \geq C|\xi|^m, \quad |\xi| \geq R.$$

Let $\eta \in \mathbb{R}^n$ be such that $P_m(\eta) = 0$. Suppose that $\eta \neq 0$. Since

$$P_m(\eta) = |\eta|^m P_m(\eta'),$$

we see that $P_m(\eta') = 0$. But for all positive numbers t with $t \geq R$,

$$t^m |P_m(\eta')| = |P_m(t\eta')| \geq Ct^m.$$

Therefore

$$|P_m(\eta')| \geq C > 0$$

and this is a contradiction. Conversely, for all ξ' in $\mathbb{R}^n$ with $|\xi'| = 1$, we get

$$P_m(\xi') \neq 0.$$

Since $\{\xi' \in \mathbb{R}^n : |\xi'| = 1\}$ is compact, it follows that there is a positive constant C such that

$$|P_m(\xi')| \geq C, \quad |\xi'| = 1.$$

Thus, there exists a positive constant R such that

$$|P_m(\xi)| = |\xi|^m |P_m(\xi')| \geq C(1 + |\xi|)^m, \quad |\xi| \geq R.$$

Therefore $P(D)$ is elliptic.

10.2. We assume that $\sigma \in S^{m_1}$ and $\tau \in S^{m_2}$. Then $T_\sigma T_\tau = T_\lambda$, where

$$\lambda \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu \sigma)(\partial_x^\mu \tau).$$

Thus, $\lambda - \sigma\tau \in S^{m_1+m_2-1}$. So, there exist positive constants C, C' and R such that for all ξ with $|\xi| > R$,

$$\begin{aligned} |\lambda(x,\xi)| &= |\sigma(x,\xi)\tau(x,\xi) + \lambda(x,\xi) - \sigma(x,\xi)\tau(x,\xi)| \\ &\geq C(1+|\xi|)^{m_1+m_2} - |\lambda(x,\xi) - \sigma(x,\xi)\tau(x,\xi)| \\ &\geq C(1+|\xi|)^{m_1+m_2} - C'(1+|\xi|)^{m_1+m_2-1} \\ &= C(1+|\xi|)^{m_1+m_2} \left\{ 1 - \frac{C'}{C}(1+|\xi|)^{-1} \right\}. \end{aligned}$$

Thus, there is a positive constant R' such that

$$|\lambda(x,\xi)| \geq \frac{C}{2}(1+|\xi|)^{m_1+m_2}, \quad |\xi| \geq R'.$$

Therefore $T_\sigma T_\tau$ is elliptic.

10.3. We suppose that $\sigma \in S^m$. Then $T_\sigma^* = T_\tau$, where $\tau \in S^m$ and

$$\tau \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma}.$$

So, $\tau - \overline{\sigma} \in S^{m-1}$. Therefore there exist positive constants C, C' and R such that

$$\begin{aligned} |\tau(x,\xi)| &= |\overline{\sigma}(x,\xi) + \tau(x,\xi) - \overline{\sigma}(x,\xi)| \\ &\geq C(1+|\xi|)^m - C'(1+|\xi|)^{m-1} \\ &= C(1+|\xi|)^m \left\{ 1 - \frac{C}{C'}(1+|\xi|)^{-1} \right\} \end{aligned}$$

for all ξ with $|\xi| \geq R$. Thus, there is a positive constant R' such that

$$|\tau(x,\xi)| \geq \frac{C}{2}(1+|\xi|)^m, \quad |\xi| \geq R'.$$

Thus, T_σ^* is elliptic.

10.4. Let T_σ be an elliptic pseudo-differential operator with parametrices T_τ and $T_{\tau'}$. Then there exist infinitely smoothing operators R and S such that

$$T_\tau T_\sigma = I + R,$$

and

$$T_\sigma T_{\tau'} = I + S.$$

Therefore

$$T_\tau T_\sigma T_{\tau'} = T_{\tau'} + R T_{\tau'}$$

and hence

$$T_\tau (I + S) = T_{\tau'} + R T_{\tau'}.$$

Thus,

$$T_\tau - T_{\tau'} = R T_{\tau'} - T_\tau S$$

and we are done.

10.5. The answer is yes. Indeed, let T_τ be a parametrix of an elliptic pseudo-differential operator T_σ. Then T_σ is a parametrix of T_τ. Thus, T_τ is elliptic.

10.10. Let $P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ be an elliptic linear partial differential operator with constant coefficients. Its symbol is

$$P(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n.$$

By ellipticity, there exist positive constants C and R such that

$$|P(\xi)| \ge C(1+|\xi|)^m, \quad |\xi| \ge R.$$

Let $\psi \in C^\infty(\mathbb{R}^n)$ be such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \ge 2R, \\ 0, & |\xi| \le R. \end{cases}$$

Let τ be the function on $\mathbb{R}^n$ such that

$$\tau(\xi) = \begin{cases} \frac{\psi(\xi)}{P(\xi)}, & |\xi| \ge R, \\ 0, & |\xi| \le R. \end{cases}$$

Then $\tau \in S^{-m}$ and

$$T_\tau T_P = T_{\tau P}.$$

But

$$\tau(\xi)P(\xi) = \begin{cases} \frac{\psi(\xi)}{P(\xi)}, & |\xi| \ge R, \\ 0, & |\xi| < R. \end{cases}$$

Let φ be the function on $\mathbb{R}^n$ defined by

$$\varphi(\xi) = 1 - \psi(\xi), \quad \xi \in \mathbb{R}^n.$$

Then $\varphi \in C_0^\infty(\mathbb{R}^n)$ and

$$\tau(\xi)P(\xi) = 1 - \varphi(\xi), \quad \xi \in \mathbb{R}^n.$$

Therefore

$$T_\tau T_P = I + T_{-\varphi}.$$

Since $\varphi \in C_0^\infty(\mathbb{R}^n)$, it follows that $T_{-\varphi}$ is infinitely smoothing. Therefore T_τ is a left parametrix and hence a parametrix.