A Brief Summary of Laurent Series

Let f be a holomorphic function on an annulus $D = \{ z \in \mathbb{C} : r < |z - z_0| < R \}$. Then f can be expanded into a series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n},$$

where both series converge on D and uniformly on every subannulus of D concentric with D. Let C be any simple closed contour in D ~~concentric with~~ containing z_0 and oriented once in the counterclockwise direction.

Then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta, \quad n = 0, \pm 1, \pm 2, \ldots .$$

We call $\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$ the Laurent series of f at z_0.

<u>Theorem</u>: Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$ be series such that

- $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges on $\{ z \in \mathbb{C} : |z - z_0| < R \}$,
- $\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$ converges on $\{ z \in \mathbb{C} : |z - z_0| > r \}$,
- $r < R$.

Then there exists a unique holomorphic function f on $D = \{ z \in \mathbb{C} : r < |z - z_0| < R \}$ such that the Laurent series of f on D is

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}, \quad r < |z - z_0| < R.$$

Proof: Assume $z_0 = 0$. Let $\zeta = \frac{1}{z}$. Then

$$\sum_{n=1}^{\infty} a_{-n}\zeta^n$$

converges on $\{\zeta \in \mathbb{C} : |\zeta| < \frac{1}{r}\}$. Let

$$H(\zeta) = \sum_{n=1}^{\infty} a_{-n}\zeta^n, \quad |\zeta| < \frac{1}{r}.$$

Then H is holomorphic on $\{\zeta \in \mathbb{C} : |\zeta| < \frac{1}{r}\}$. So the function $h(z) = H\left(\frac{1}{z}\right)$ is holomorphic on $|z| > r\}$.

$$\text{And } h(z) = \sum_{n=1}^{\infty} a_{-n} z^{-n}, \quad |z| > r.$$

Now the function $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, is holomorphic. So the function f given by

$$f(z) = g(z) + h(z)$$

is holomorphic on $D = \{z \in \mathbb{C} : r < |z| < R\}$. So $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n}$ is the Laurent series of f in D at $z_0 = 0$. To do this, let C be a simple closed contour in D enclosing 0 and oriented once in the counterclockwise direction. Then for all $j = 0, \pm 1, \pm 2, \ldots$,

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{j+1}}\,dz = \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} a_n z^{n-j-1}\,dz$$

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$$= \frac{1}{2\pi i} a_j\, 2\pi i = a_j.$$

Example: Find the Laurent series of
$$f(z) = \frac{z^2-2z+3}{z-2}$$
on $\{z \in \mathbb{C} : |z-1| > 1\}$.

Solution: $z_0 = 1$, $r = 1$, $R = \infty$. $\therefore$ f is holomorphic on $\{z \in \mathbb{C} : 1 < |z-1| < \infty\}$.

Now
$$\frac{1}{z-2} = \frac{1}{z-1-1} = \frac{1}{z-1}\,\frac{1}{1-\frac{1}{z-1}}$$
$$= \frac{1}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n} = \sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}.$$

But
$$z^2-2z+3 = z^2-2z+1+2 = (z-1)^2+2.$$

$\therefore$ $f(z)$
$$= \left\{(z-1) + 1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \cdots\right\} + \left\{\frac{2}{z-1} + \frac{2}{(z-1)^2} + \cdots\right\}$$
$$= (z-1) + 1 + \sum_{n=1}^{\infty}\frac{3}{(z-1)^n}, \quad |z-1| > 1.$$

Example: Find the Laurent series of $e^{\frac{1}{z}}$ at $z=0$.

Solution:
$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n}$$
$$= 1 + \sum_{n=1}^{\infty}\frac{1}{n!}z^{-n}, \quad z \neq 0.$$

Why Laurent Series?

Isolated Singularities

Definition: Let $\omega = f(z)$ be a complex-valued function.

Let $z_0 \in \mathbb{C}$ be $\ni$ $\begin{cases} f \text{ is not holomorphic at } z_0 \\ f \text{ is holomorphic on some punctured disk of } z_0. \end{cases}$

Then we say that z_0 is an isolated singularity of f.

Let z_0 be an isolated singularity of f. Then f is holomorphic on a punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$.

Three Possibilities on $\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$ of f

- If $a_{-n} = 0$ for $n = 1, 2, \dots$, then we call z_0 a <u>removable singularity</u> of f.
- If $a_{-m} \neq 0$ for some $m \in \mathbb{N}$ and $a_{-n} = 0$ for all $n > m$, then we call z_0 a <u>pole of order</u> m of f. A pole of order 1 is called a <u>simple pole</u>.
- If $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$, then we call z_0 an <u>essential singularity</u> of f.

Example: Classify the isolated singularities of $f(z) = e^{\frac{1}{z}}$, $z \neq 0$

Solution 0 is the only isolated singularity of f.

Also,
$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots \qquad z \neq 0$$
the Laurent series of $e^{\frac{1}{z}}$

Now note that $a_{-n} = \frac{1}{n!} \neq 0$ for all $n \in \mathbb{N}$.