Answers to Assignment 5

11.1. We see that $\operatorname{Log} 1 = 0$ and

$$\frac{d^n}{dz^n}(\operatorname{Log} 1) = (-1)^{n+1}(n-1)!, \quad n = 1, 2, \ldots.$$

Thus, the Taylor series of $\operatorname{Log} z$ at 1 is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n-1)!}{n!}(z-1)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}(z-1)^n$$

and the largest disk of convergence is $\{z \in \mathbb{C} : |z-1| < 1\}$.

11.2. Since for all $w \in \mathbb{C}$, the Maclaurin series of e^w is

$$\sum_{n=0}^{\infty}\frac{1}{n!}w^n,$$

it follws that e^{z^2} has Maclaurin series

$$\sum_{n=0}^{\infty}\frac{1}{n!}z^{2n}.$$

11.3. Write the power series as

$$\sum_{n=0}^{\infty} n^2 z^n = z^2\sum_{n=0}^{\infty} n(n-1)z^{n-2} + z\sum_{n=0}^{\infty} nz^{n-1}.$$

But for $|z| < 1$

$$\begin{aligned}
&\sum_{n=0}^{\infty} n(n-1)z^{n-2} \\
= &\ \frac{d^2}{dz^2}\sum_{n=0}^{\infty} z^n \\
= &\ \frac{d^2}{dz^2}\left(\frac{1}{1-z}\right) \\
= &\ \frac{d}{dz}\left(\frac{1}{(1-z)^2}\right) \\
= &\ \frac{2}{(1-z)^3}.
\end{aligned}$$

We have seen that for $|z| < 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} nz^{n-1} &= \frac{d}{dz}\sum_{n=0}^{\infty} z^n \\
&= \frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2}.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} n^2 z^n = \frac{2z^2}{(1-z)^3} + \frac{z}{(1-z)^2}$$

and the radius of convergence is 1.

11.4. Since

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|^n} == \inf_{k\geq 1} \sup_{n\geq k} \sqrt[n]{|a_n|^n} = 2.$$

Using Hadamard's formula,

$$R = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|^n}} = \frac{1}{2}.$$

11.5. Separating the evens from the odds, we have for $|z| < \frac{1}{2}$,

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} 2^{2n} z^{2n} + \sum_{n=0}^{\infty} z^{2n+1}.$$

But

$$\sum_{n=0}^{\infty} 2^{2n} z^{2n} = \sum_{n=0}^{\infty} (4z^2)^n = \frac{1}{1-4z^2}.$$

And

$$\sum_{n=0}^{\infty} z^{2n+1} = z \sum_{n=0}^{\infty} z^{2n} = \frac{z}{1-z^2}.$$

Therefore

$$\sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-4z^2} + \frac{z}{1-z^2}.$$