Answers to Assignment 4

9.1. Let Γ be a closed contour in $\mathbb{C}$. Since e^{z^2} is holomorphic on $\mathbb{C}$ and $\mathbb{C}$ is simply connected, it follows from Cauchy's integral theorem that

$$\int_\Gamma e^{z^2} dz = 0.$$

By Theorem 8.11, e^{z^2} has an antiderivative on $\mathbb{C}$.

9.2 Let $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ be oriented once in the counterclockwise direction. Paramietrize it by

$$z = e^{it}, \quad 0 \leq t \leq 2\pi.$$

Then

$$\int_\Gamma \overline{z} dz = \int_0^{2\pi} e^{-it} i e^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

So, it is not true that $\int_\Gamma \overline{z} dz = 0$ for every closed contour Γ in $\mathbb{C}$. This does not contradict Cauchy's integral theorem because $\overline{z}$ is not holomorphic on $\mathbb{C}$.

9.3. The function $\mathrm{Log}(z+3)$ is holomorphic at all z with $z + 3 \notin (-\infty, 0]$. Let D be the open disk with center 0 and radius 2.5. Then $\mathrm{Log}(z+3)$ is holomorphic on D (Why?) and D is simply connected. Therefore

$$\int_C \mathrm{Log}(z+3)\, dz = 0$$

by Cauchy's integral theorem.

9.4. By partial fraction decomposition,

$$\begin{aligned}
\frac{2z^2 - z + 1}{(z-1)^2(z+1)} &= \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2} \\
&= \frac{A(z-1)^2 + B(z-1)(z+1) + C(z+1)}{(z-1)^2(z+1)} \\
&= \frac{A(z^2 - 2z + 1) + B(z^2 - 1) + C(z+1)}{(z-1)^2(z+1)}.
\end{aligned}$$

Therefore

$$\begin{aligned} A+B &= 2 \\ -2A \quad +C &= -1 \\ A-B+C &= 1. \end{aligned}$$

So, $A=B=C=1$. Therefore

$$\int_\Gamma \frac{2z^2-z+1}{(z-1)^2(z+1)}dz = \int_\Gamma \left(\frac{1}{z+1}+\frac{1}{z-1}+\frac{1}{(z-1)^2}\right)dz.$$

Let Γ_l amd Γ_r be the contours on the left and on the right, respectively. Γ_l can be continuously deformed into a circle C_{-1} centered at -1 and lying inside Γ_l and Γ_r can be continuously deformed into a circle C_1 centered at 1 and lying inside Γ_r. Now, by Cauchy's integral theorem,

$$\int_{\Gamma_l} \left(\frac{1}{z+1}+\frac{1}{z-1}+\frac{1}{(z-1)^2}\right)dz = \int_{C_{-1}} \frac{1}{z+1}dz = 2\pi i.$$

By Cauchy's integral theorem again,

$$\int_{\Gamma_r} \left(\frac{1}{z+1}+\frac{1}{z-1}+\frac{1}{(z-1)^2}\right)dz = \int_{C_1} \frac{1}{z-1}dz + \int_{C_1} \frac{1}{(z-1)^2}dz = -2\pi i.$$

Thus,

$$\int_\Gamma \frac{2z^2-z+1}{(z-1)^2(z+1)}dz = 2\pi i - 2\pi i = 0.$$

10.1. Let C be the unit citcle with center at the origin and oriented once in the counterclockwise direction. Let $f(z) = \cos z$.

$$f^{(27)}(0) = \frac{(27)!}{2\pi}\int_C \frac{\cos z}{z^{28}}dz.$$

Now,

$$\begin{aligned}
&f(0) = \cos\, 0 = 1, \\
&f'(0) = -\sin\, 0 = 0, \\
&f''(0) = -\cos\, 0 = -1, \\
&f'''(0) = \sin\, 0 = 0, \\
&f^{(4)}(0) = \cos\, 0 = 1, \\
&... \\
&f^{(7)}(0) = 0, \\
&... \\
&f^{(11)}(0) = 0, \\
&... \\
&f^{(15)}(0) = 0, \\
&... \\
&f^{(19)}(0) = 0, \\
&... \\
&f^{(23)}(0) = 0, \\
&... \\
&f^{(27)}(0) = 0.
\end{aligned}$$

Therefore

$$\int_C \frac{\cos\, z}{z^{28}} dz = 0.$$

Next, write

$$\int_C \left(\frac{z-2}{2z-1}\right)^3 dz = \int_C \frac{(z-2)^3/8}{(z-(1/2))^3} dz.$$

Let $f(z) = \frac{(z-2)^3}{8}$. Then

$$f'(z) = \frac{3}{8}(z-2)^2$$

and

$$f''(z) = \frac{3}{4}(z-2).$$

So,

$$f''\left(\frac{1}{2}\right) = -\frac{9}{16}.$$

Using Cauchy's integral theorem, we get

$$\int_C \left(\frac{z-2}{2z-1}\right)^3 dz = -\frac{9\pi i}{16}.$$

10.2. Write Γ as

$$\Gamma_+ + \Gamma_-,$$

where Γ_+ is the upper semicicle + $[-3, 3]$ oriented once in the counterclockwise direction and Γ_- is the lower semicircle + the line segment from 3 to -3 oriented once in the counterclockwise direction. Then

$$\int_{\Gamma_-} \frac{e^{iz}}{(z^2+1)^2} dz = \int_{\Gamma_-} \frac{e^{iz}/(z-i)^2}{(z+i)^2} dz = 2\pi i f_-'(-i),$$

where

$$f_-(z) = \frac{e^{iz}}{(z-i)^2}.$$

Therefore

$$f_-'(-i) = \left.\frac{(z-i)^2 ie^{iz} - 2e^{iz}(z-i)}{(z-i)^4}\right|_{-i} = \frac{-4ie + 4ei}{16} = 0.$$

Next,

$$\int_{\Gamma_+} \frac{e^{iz}}{(z^2+1)^2} dz = \int_{\Gamma_+} \frac{e^{iz}/(z+i)^2}{(z-i)^2} dz = 2\pi i f_+'(i),$$

where

$$f_+(z) = \frac{e^{iz}}{(z+i)^2}.$$

Therefore

$$f_+'(i) = \left.\frac{(z+i)^2 ie^{iz} - 2e^{iz}(z+i)}{(z+i)^4}\right|_{i} = \frac{-4ie^{-1} - 4ie^{-1}}{-4} = 2ie^{-1}.$$

Therefore

$$\int_{\Gamma} \frac{e^{iz}}{(z^2+1)^2} dz = 2ie^{-1}.$$

10.3. By Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0} dz,$$

where C_r is the circle $\{z \in \mathbb{C} : |z - z_0| = r\}$ oriented once in the counterclockwise direction. Parametrizing C_r by

$$z = z_0 + re^{i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Then

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{i\theta}) re^{i\theta} i re^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

10.4. By Cauchy's integral formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C_r and its parametrization are as in 10.3. Then

$$f^{(n)}(z_0) \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{r^{n+1} e^{i(n+1)\theta}} i re^{i\theta} d\theta = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta.$$

10.5. Using polar coordinates,

$$\begin{aligned} \int_{|z| \leq 1} f(x + iy) dx\, dy &= \int_0^{2\pi} \int_0^1 f(re^{i\theta}) r dr\, d\theta \\ &= \int_0^1 \left(\int_0^{2\pi} f(re^{i\theta})\, d\theta \right) dr. \end{aligned}$$

By the Mean Value Property of holomorphic functions,

$$\int_0^{2\pi} f(re^{i\theta}) d\theta = 2\pi f(0).$$

Therefore

$$\int_{|z|\leq 1} f(x+iy)dx\,dy = 2\pi f(0)\int_0^1 r\,dr = \pi f(0).$$

10.6. Orienting C_R once in the counterclockwise direction, we get by Cauchy's integral formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad n = 0, 1, 2, \ldots.$$

On C_R,

$$\left|\frac{f(z)}{z-z_0}\right| \leq \frac{M}{R^{n+1}}.$$

So, by the ML-theorem,

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{2\pi R^{n+1}}2\pi R = \frac{Mn!}{R^n}, \quad n = 0, 1, 2, \ldots.$$

10.7. Let f be a bounded and entire function. Then there exists a positive constant M such that

$$|f(z)| \leq M, \quad z \in \mathbb{C}.$$

By Cauchy's estimate with $n = 1$, we get

$$|f'(z)| \leq \frac{M}{R}$$

for every positive number M. Ler $R \to \infty$. Then

$$f'(z) = 0, \quad z \in \mathbb{C}.$$

Therefore f is a constant function.

10.8. Suppose by way of contradiction that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $\frac{1}{P}$ is an enire function. Since $\frac{1}{P(z)} \to 0$ as $|z| \to \infty$, it follows that $\frac{1}{P}$ is a bounded function on $\mathbb{C}$. By Liouville's theorem, $\frac{1}{P}$ is a constant function. Therefore P is a constant function. This contradicts the assumption that P is a polynomial.