Answers to Test 1

1.(a) Let $u(x,y) = 2xy$ and $v(x,y) = x^2 + y^2$ for all $z = x + iy \in \mathbb{C}$. Then

$$\frac{\partial u}{\partial x} = 2y,$$

$$\frac{\partial v}{\partial y} = 2y,$$

$$\frac{\partial u}{\partial y} = 2x$$

and

$$\frac{\partial v}{\partial x} = 2x$$

for all $z = x + iy \in \mathbb{C}$. Therefore

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

if and only if $x = 0$, i.e., for all z in the y-axis.
(a) f is differentiable at all points in the y-axis.
(b) f is nowhere holomorphic.

2.(a) Since $|1 + \sqrt{3}i| = 2$ and $\mathrm{Arg}(1 + \sqrt{3}i) = \frac{\pi}{3}$, we have

$$\mathrm{Log}(1 + \sqrt{3}i) = \ln|1 + \sqrt{3}i| + i\mathrm{Arg}(1 + \sqrt{3}i) = \ln 2 + \frac{\pi}{3}i.$$

(b) Let $\tau = 0$. Then $f(z) = \log_0(2z - 3)$ is holomorphic at all z unless $2z - 3 \in [0, \infty)$. At $z = 1$, we have $2(1) - 3 = -1 \in [0, \infty)$. Therefore $f(z) = \log_0(2z - 3)$ is holomorphic at $z = 1$.

3. Let $u(x,y) = e^{-y}\cos x$ and $v(x,y) = e^{-y}\sin x$ for all $z = x + iy \in \mathbb{C}$. Then

$$\frac{\partial u}{\partial x} = -e^{-y}\sin x,$$

$$\frac{\partial v}{\partial y} = -e^{-y}\sin\, x,$$
$$\frac{\partial u}{\partial y} = -e^{-y}\cos\, x$$

and

$$\frac{\partial v}{\partial x} = e^{-y}\cos\, x$$

for all $z = x + iy \in \mathbb{C}$. Then for all $z = x + iy \in \mathbb{C}$,

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases}$$

Therefore f is an entire function.

4. Suppose by way of contradiction that f is holomorphic at z_0 in $\mathbb{C}$. Then there exists a neighborhood N of z_0 such that f is holomorphic on N. Since $\arg_\tau$ is a real-valued function on N, it follows from a previous exercise (Exercise 5.6) that f is a constant function on N. This is a contradiction.

5. Let C be parametrized by

$$z = e^{it}, \quad 0 \le t \le 2\pi.$$

Then

$$\int_C \overline{z}dz = \int_0^{2\pi} e^{-it} ie^{it}dt = i\int_0^{2\pi} dt = 2\pi i.$$

6. We use the ML-theorem. To find M, we note that for all $z \in \mathbb{C}$,

$$|e^{\cos\, z}| = e^{\mathrm{Re}\,(\cos\, z)}.$$

But for $z = x + iy$,

$$\mathrm{Re}\,(\cos\, z) = \mathrm{Re}\frac{e^{iz} + e^{-iz}}{2}$$

$$
\begin{aligned}
&= \mathrm{Re}\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
&= \mathrm{Re}\frac{e^{(ix-y)} + e^{(-ix+y)}}{2} \\
&= \mathrm{Re}\frac{e^{-y}(\cos\, x + i\sin\, x) + e^{y}(\cos\, x - i\sin\, x)}{2} \\
&= \frac{e^{-y}\cos\, x + e^{y}\cos\, x}{2}.
\end{aligned}
$$

So, for all $z = x + iy \in \Gamma$, $y = 0$, Therefore for all $z \in \Gamma = [0, 1]$,

$$|e^{\cos\, z}| \leq e^{1}.$$

Since $L = 1$, we have

$$\left|\int_{\Gamma} e^{\cos\, z} dz\right| \leq e.$$