Answers to Assignment 3

7.1. Let $f(z) = \operatorname{Log}(z^2+1)$. Then f is holomorphic at all z with $z^2+1 \notin (-\infty, 0]$. Since $(-1)^2+1 = 2 \notin (-\infty, 0]$, it follows that f is holomorphic at -1. Moreover,

$$f'(-1) = \left.\frac{2z}{z^2+1}\right|_{z=-1} = -1.$$

7.2. Suppose that by way of contradiction that there exists a holomorphic function F on D such that

$$F'(z) = \frac{1}{z}, \quad z \in D.$$

Then for all $z \in D - (-2, -1)$,

$$F(z) = \operatorname{Log} z + C,$$

where C is a complex constant. Let $z_0 \in (-2, -1)$. Then for all $z \in D$ with $z \to z_0$ from above the cut,

$$F(z) \to \ln|z_0| + i\pi + C.$$

For all $z \in D$ with $z \to z_0$ from below the cut,

$$F(z) \to \ln|z_0| - i\pi + C.$$

Therefore F is not continuous at z_0. This contradicts the holomorphicity of F on D.

7.3. Let f be the principal branch. Then

$$f(z) = e^{(1+i)\operatorname{Log} z}$$

is holomorphic at all points z not in $(-\infty, 0]$. So, f is holomorphic at i. Furthermore,

$$\begin{aligned}
f'(i) &= e^{(1+i)\operatorname{Log} z}(1+i)\frac{1}{z}\bigg|_{z=i} \\
&= e^{(1+i)\operatorname{Log} i}\frac{1+i}{i} \\
&= e^{(1+i)(\ln|i|+i\operatorname{Arg} i)}\frac{1+i}{i} \\
&= e^{(1+i)i\pi/2}\frac{1+i}{i} \\
&= e^{-\pi/2}e^{i\pi/2}\frac{1+i}{i} \\
&= e^{-\pi/2}(1+i).
\end{aligned}$$

7.4.(a) We want to to find a branch $w = f(z)$ such that f is holomorphic on $\{z \in \mathbb{C} : |z| < 1\}$ and

$$w^2 = z^2 - 1 = z^2\left(1 - \frac{1}{z^2}\right).$$

Let

$$w = z\left(1 - \frac{1}{z^2}\right)^{1/2} = ze^{\frac{1}{2}\log_0\left(1-\frac{1}{z^2}\right)}.$$

Then $w = f(z)$ is holomorphic at all z unless $1 - \frac{1}{z^2} \in [0, \infty)$. Then

$$\begin{aligned}
1 - \frac{1}{z^2} \in [0, \infty) &\Leftrightarrow 1 - \frac{1}{z^2} \geq 0 \\
&\Leftrightarrow \frac{1}{z^2} \leq 1 \\
&\Leftrightarrow z^2 \geq 1 \\
&\Leftrightarrow |z|^2 \geq 1 \\
&\Leftrightarrow |z| \geq 1.
\end{aligned}$$

Therefore $w = f(z)$ is holomorphic on $\{z \in \mathbb{C} : |z| < 1\}$.

(b) See Example 7.9 in the textbook.

(c) Let $\mathbb{C}^*$ be the complex plane $\mathbb{C}$ cut along the imaginary axis from $-i$ to i. We want to find a branch $w = f(z)$ such that f is holomorphic on $\mathbb{C}^*$ and

$$w^2 = (z^2 + 1) = z^2 \left(1 + \frac{1}{z^2}\right).$$

Let

$$w = z \left(1 + \frac{1}{z^2}\right)^{1/2} = e^{\frac{1}{2}\mathrm{Log}\left(1+\frac{1}{z^2}\right)}.$$

Then w is holomorphic at all z unless $1 + \frac{1}{z^2} \leq 0$ or equivalently $\frac{1}{z^2} \leq -1$. Let $z = \alpha i$ with $\alpha \neq 0$. Then $z^2 = -\alpha^2$. So,

$$\begin{aligned} \frac{1}{z^2} \leq -1 \quad &\Leftrightarrow \quad -\frac{1}{\alpha^2} \leq -1 \\ &\Leftrightarrow \quad \alpha^2 \leq 1 \\ &\Leftrightarrow \quad -1 \leq \alpha \leq 1. \end{aligned}$$

So, f is holomorphic on the y-axis cut along $-i$ and i. Let $z = x \in \mathbb{R}$ with $x \neq 0$. Then

$$\frac{1}{z^2} = \frac{1}{x^2} > -1.$$

Then f is holomorphic at all points $z \in \mathbb{R} - \{0\}$. Finally, let z be neither real nor imaginary. Then $\frac{1}{z^2}$ cannot be ≤ -1. Therefore f is holomorphic on $\mathbb{C}^*$.

(d) We want to find a branch $w = f(z)$ such that f is holomorphic on $\{z \in \mathbb{C} : |z| > 1\}$ and

$$w^5 = z^5 - 1 = z^5 \left(1 - \frac{1}{z^5}\right)$$

Let

$$w = z \left(1 - \frac{1}{z^5}\right)^{1/5} = z e^{\frac{1}{5}\mathrm{Log}\left(1-\frac{1}{z^5}\right)}.$$

Then f is holomorphic at all z unless $\frac{1}{z^5} \geq 1$, which is the same as $|z| \leq 1$. Therefore f is holomorphic on $\{z \in \mathbb{C} : |z| > 1\}$.

7.5. Suppose by way of contradiction that the function $w = f(z) = \arg_\tau z$ is holomorphic at some point z_0 in $\mathbb{C}_\tau^\circ$. Then there exists a neighborhood N of z_0 such that f is holomorphic on N. Since $f(z)$ is real-valued, it follows that f is a constant function on N. This is a contradiction.

8.1. Let C be the unit circle centered at the origin and oriented once in the counterclockwise direction. Let it be parametrized by

$$z = e^{it}, \quad 0 \leq t \leq 2\pi.$$

Then

$$\begin{aligned}
\int_C \frac{1}{z}\left(z + \frac{1}{z}\right)^{2n} dz &= \int_0^{2\pi} e^{-it}(e^{it} + e^{-it}) ie^{it} dt \\
&= i \int_0^{2\pi} (2\cos t)^{2n} dt.
\end{aligned}$$

Using the binomial theorem,

$$\left(z + \frac{1}{z}\right)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} z^k \left(\frac{1}{z}\right)^{2n-k} = \sum_{k=0}^{2n} \binom{2n}{k} z^{2k-2n}.$$

So,

$$\begin{aligned}
\int_C \frac{1}{z}\left(z + \frac{1}{z}\right)^{2n} dz &= \int_C \frac{1}{z}\left(\sum_{k=0}^{2n} \binom{2n}{k} z^{2k-2n}\right) dz \\
&= \sum_{k=0}^{2n} \binom{2n}{k} \int_C z^{2k-2n-1} dz \\
&= \binom{2n}{n} \int_C z^{-1} dz \\
&= \binom{2n}{n} 2\pi i.
\end{aligned}$$

Therefore

$$i \int_0^{2\pi} (2 \cos t)^{2n} dt = \binom{2n}{n} 2\pi i,$$

which is the same as

$$\frac{1}{2\pi} \int_0^{2\pi} (2 \cos t)^{2n} dt = \frac{(2n)!}{(2n-n)!n!} = \frac{(2n)!}{(n!)^2}.$$

8.2. For all $z \in \Gamma$,

$$|\mathrm{Log}\, z| = |\ln |z| + i\mathrm{Arg}\, z| = |\ln 1 + i\mathrm{Arg}\, z| = |\mathrm{Arg}\, z| \leq \frac{\pi}{2}.$$

Therefore by the ML-theorem,

$$\left| \int_\Gamma \mathrm{Log}\, z \, dz \right| \leq \frac{\pi}{2}\frac{\pi}{2} = \frac{\pi^2}{4}.$$

8.3. For all $z = x + iy \in \Gamma$,

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= \frac{e^{ix-y} - e^{-ix+y}}{2i} \\ &= \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x}{2i}) \end{aligned}$$

So, for all $z \in \Gamma$,

$$\mathrm{Re}(\sin z) = \frac{e^{-y} \sin x - e^{y} \sin x}{2} = 0.$$

Therefore for all $z \in \Gamma$,

$$|e^{\sin z}| = e^{\mathrm{Re}\, \sin z} = 1.$$

So, by the ML-theorem,

$$\left| \int_\Gamma e^{\sin z} dx \right| \leq 1.$$

8.4. Let $\tau \in \mathbb{R}$ be such that C lies inside the cut plane $\mathbb{C}_\tau^\circ$. Then $\frac{1}{z}$ has an antiderivative, namely, $\arg_\tau z$. Therefore by Theorem 8.11 in the textbook,

$$\int_C \frac{1}{z} dz = 0.$$