Answers to Assignment 2

To the TA and Students of MATH 3410 3.0: Pay special attention to Question 2.

5.1. Let $u(x,y) = x^3 - y^3$ and $v(x,y) = -3xy$ for all x and y in $\mathbb{R}$. Then

$$\frac{\partial u}{\partial x} = 3x^2,$$

$$\frac{\partial v}{\partial y} = -3x,$$

$$\frac{\partial u}{\partial y} = 3y^2$$

and

$$\frac{\partial v}{\partial x} = -3y$$

for all x and y in $\mathbb{R}$. Therefore

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

if and only if $x^2 = -x$ and $y^2 = y$. So, the Cauchy–Riemann equations are satisfied at the points $(-1,0)$, $(0,0)$, $(-1,1)$ and $(0,1)$. Thus, f is differentiable at only these four points and is nowhere holomorphic.

5.2. Let

$$w = f(z) = \begin{cases} \frac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{x^2+y^2}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Let

$$u(x,y) = \begin{cases} \frac{x^{4/3}y^{5/3}}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Let

$$v(x,y) = \begin{cases} \frac{x^{5/3}y^{4/3}}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Then

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x^{4/3}0}{x^3} = 0.$$

Similarly, $\frac{\partial v}{\partial y}(0,0)$, $\frac{\partial u}{\partial y}(0,0)$ and $\frac{\partial v}{\partial x}(0,0)$ are all equal 0. Therefore u and v satisfy the Cauchy–Riemann equations at $(0,0)$. To see that f is not differentiable at $z=0$, we note that

$$\begin{aligned} & \lim_{z\to 0} \frac{f(z)-f(0)}{z} \\ = & \lim_{z\to 0} \frac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{(x^2+y^2)(x+iy)}. \end{aligned}$$

Let $z \to 0$ along the line $y = x$. Then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{x^3+ix^3}{2x^2(x+ix)} = \frac{1}{2}.$$

Let $z \to 0$ along the line $y = -x$. Then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{-x^3+ix^3}{2x^2(x-ix)} = -\frac{1}{2}.$$

Therefore $f'(0)$ does not exist. So, f is not differentiable at $z=0$.

5.3. Writing

$$w = f(z) = u(x,y) + iv(x,y), \quad z = x+iy \in \mathbb{C}.$$

Suppose that f is holomorphic on G. Then

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \end{cases}$$

So, for all $z \in G$,

$$\begin{aligned}(\overline{\partial} f)(z) &= \frac{\partial}{\partial x}(u+iv) + i\frac{\partial}{\partial y}(u+iv) \\ &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = 0.\end{aligned}$$

Therefore $\overline{\partial} f)(z) = 0$ for all $z \in G$. The converse is the same by simply reversing the argument.

5.4. If

$$f(z) = u(x,y) + iv(x,y), \quad (x,y) \in G,$$

then

$$\overline{f(\overline{z})} = U(x,y) + iV(x,y), \quad (x,y) \in G,$$

where

$$U(x,y) = u(x,-y)$$

and

$$V(x,y) = -v(x,-y)$$

for all $(x,y) \in G$. Therefore

$$\frac{\partial U}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,-y),$$

$$\frac{\partial V}{\partial y}(x,y) = \frac{\partial v}{\partial y}(x,-y),$$

$$\frac{\partial U}{\partial y}(x,y) = \frac{\partial u}{\partial y}(x,-y)$$

and

$$\frac{\partial V}{\partial x}(x,y) = -\frac{\partial v}{\partial x}(x,-y)$$

for all $(x,y) \in G$. Since f is holomorphic on G and G is symmetric, we get

$$\begin{cases} \frac{\partial u}{\partial x}(x,-y) = \frac{\partial v}{\partial y}(x,-y), \\ \frac{\partial v}{\partial x}(x,-y) = -\frac{\partial u}{\partial y}(x,-y), \end{cases}$$

for all $(x, y) \in G$. Therefore

$$\begin{cases} \frac{\partial U}{\partial x}(x,y) = \frac{\partial V}{\partial y}(x,y), \\ \frac{\partial V}{\partial x}(x,y) = -\frac{\partial U}{\partial y}(x,y), \end{cases}$$

for all $(x, y) \in G$. Therefore $w = \overline{f(\overline{z})}$ is holomorphic on G.

5.5. Let

$$f(z) = u(x,y) + iv(x,y), \quad (x,y) \in D.$$

Then

$$0 = f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y), \quad (x,y) \in D.$$

By the Cauchy–Riemann equations,

$$0 = f'(z) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y), \quad (x,y) \in D.$$

Therefore on D,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Therefore u and v are constant functions on D. So, f is a constant function on D.

5.6. Let

$$f(z) = u(x,y) + iv(x,y), \quad (x,y) \in D.$$

Then

$$v(x,y) = 0, \quad (x,y) \in D.$$

Since f is holomorphic on D, it follows that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

for all $(x, y) \in D$. Therefore u is a constant function on D. So, $f(z) = u(x, y)$ is a constant for all $z = x + iy \in D$.

5.7. If f and $\overline{f}$ are both holomorphic on a domain D, then $f + \overline{f}$ is a real-valued holomorphic function on D. Therefore by 5.6, $f(z) = u(x, y) + iv(x, y)$ is a constant function. So, $v(v, y) = 0$ for all (x, y) in D. Using the Cauchy–Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

for all $(x, y) \in D$. Therefore $f(z) = u(x, y)$ is a constant function on D.

(**To the TA: Please discount this question. If some students can see that the statement is false, give them bonus marks.**) **Question 2: The statement is false. Here is a counterexample.** Let

$$f(z) = \cos(xy) + i \sin(xy), \quad z = x + iy \in \mathbb{C}.$$

Then

$$|f(z)| = 1, \quad z \in \mathbb{C}.$$

Hence $|f|$ is holomorphic on $\mathbb{C}$, but f is not a constant function on $\mathbb{C}$. So, 5.8 in the book cannot be made stronger. The holomorphicity of f is indispensable. Find below a proof of 5.8 in the book.

5.8. Let

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy \in D.$$

Since $|f|$ is holomorphic on D, so is $f|^2$. Since $|f|^2$ is real-valued, it follows that

$$|f(z)|^2 = u(x, y)^2 + v(x, y)^2$$

is a constant function on D. Therefore

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

and

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

Since f is holomorphic on D, the Cauchy–Riemann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for all (x, y) in D. So, on D,

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial u}{\partial y} = 0$$

and

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial u}{\partial x} = 0.$$

Multiplying the first equation by u and the second equation by v, we get after cancelling the factor 2,

$$u^2\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} = 0$$

and

$$uv\frac{\partial u}{\partial y} + v^2\frac{\partial u}{\partial x} = 0$$

for all $(x, y) \in D$. Adding the preceding two equations, we get

$$(u^2 + v^2)\frac{\partial u}{\partial x} = 0$$

for all $(x, y) \in D$. Since $u^2 + v^2$ is a constant, say,

$$u(x, y)^2 + v(x, y)^2 = C, \quad (x, y) \in D,$$

we can assume that $C \neq 0$. For otherwise

$$u(x, y)^2 + v(x, y)^2 = 0, \quad (x, y) \in D,$$

and f is then a constant function as claimed. If $C \neq 0$, then

$$\frac{\partial u}{\partial x} = 0$$

for all $(x, y) \in D$. Similarly,

$$\frac{\partial u}{\partial y} = 0$$

for all $(x, y) \in D$. Therefore u is a constant function on D. Hence v is a constant function on D. This completes the proof.

6.1. Let $z = x + iy$. Then

$$\begin{aligned} e^{e^z} &= e^{e^{x+iy}} = e^{e^x e^{iy}} \\ &= e^{e^x(\cos y + i \sin y)} = e^{e^x \cos y} e^{ie^x \sin y} \\ &= e^{e^x \cos y}(\cos(e^x \sin y) + i \sin(e^x \sin y)). \end{aligned}$$

Therefore

$$\mathrm{Re}(e^{e^z}) = e^{e^x \cos y} \cos(e^x \sin y),$$
$$\mathrm{Im}(e^{e^z}) = e^{e^x \cos y} \sin(e^x \sin y)$$

and

$$|e^{e^z}| = e^{e^x} \cos y.$$

6.2. Since

$$\begin{aligned} \sin(z + 2\pi) &= \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} \\ &= \frac{e^{iz}e^{2\pi i} - e{-}ize^{-2\pi i}}{2i} \\ &= \frac{e^{iz} - e^{-iz}}{2i} = \sin z \end{aligned}$$

for all $z \in \mathbb{C}$, $\sin z$ is a periodic function with period 2π on $\mathbb{C}$. Similarly, $\cos z$ is a periodic function of z on $\mathbb{C}$.

6.3. The answer is NO. Following the hint, we have for all $y \in \mathbb{R}$,

$$|\sin(iy)| = |\frac{e^{-y} - e^{y}}{2i}| = \frac{e^{-y} - e^{y}}{2} \rightarrow \infty$$

as $y \rightarrow \infty$.