Answers to Assignment 1

1.1. Let $z = a + ib$. Then $iz = ia - b$. Therefore $\mathrm{Re}(iz) = -b = -\mathrm{Im}\, z$.

1.2. Let $z = a + ib$. Then

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{a - ib}{a^2 + b^2}.$$

So,

$$\mathrm{Im}\left(\frac{1}{z}\right) = -\frac{b}{a^2 + b^2} < 0$$

because $b > 0$.

1.3. Let $z = a + ib$ and $w = c + id$. Then

$$z + w = (a + ib) + (c + id) = (a + c) + i(b + d) \in \mathbb{R} \Rightarrow b + d = 0.$$

Since

$$zw = (a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

is a negative real number, we have $bc + ad = 0$ and $ac - bd < 0$. So, we have

1. $b + d = 0$,
2. $bc + ad = 0$,
3. $ac - bd < 0$.

By (1) and (2), $bc - ab = 0$. Therefore $b(c - a) = 0$. So, $b = 0$ or $c = a$. If $c = a$, then by (3), $a^2 + b^2 < 0$ and this is impossible. Thus, $b = 0$ and $d = 0$. Therefore z and w must be real numbers.

1.4. Let $z = a + ib$. Then $|z| = \sqrt{a^2 + b^2}$ and $\mathrm{Re}\, z = a$. Given that $|z| = \mathrm{Re}\, z$, we have $\sqrt{a^2 + b^2} = a$. Therefore $a^2 + b^2 = a^2$ and so, $b = 0$. Thus, $z = a$ is a nonneagtive real number.

2.1.(a) $\arg(-3+i3) = \frac{3\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$.
(b) We have

$$\begin{aligned} \arg((1-i)(-\sqrt{3}+i)) &= \arg(1-i) + \arg(-\sqrt{3}+i) \\ &= \frac{7\pi}{4} + \frac{5\pi}{6} + 2k\pi = \frac{31\pi}{12} + 2k\pi, \end{aligned}$$

where $k \in \mathbb{Z}$.
(c) We have

$$\begin{aligned} \arg\left(\frac{-1+i\sqrt{3}}{2+i2}\right) &= \arg(-1+i\sqrt{3}) - \arg(2+i2) \\ &= \frac{2\pi}{3} - \frac{\pi}{4} + 2k\pi = \frac{5\pi}{12} + 2k\pi, \end{aligned}$$

where $k \in \mathbb{Z}$.

2.2.(a) $\mathrm{Arg}(-3+i3) = \frac{3\pi}{4}$.
(b) $\mathrm{Arg}((1-i)(-\sqrt{3}+i)) = \frac{7\pi}{12}$.
(c) $\mathrm{Arg}\left(\frac{-1+i\sqrt{3}}{2+i2}\right) = \frac{5\pi}{12}$.

3.1. We have

$$S = \sum_{n=0}^{100} i^n = \frac{1-i^{101}}{1-i} = \frac{1-ii^{100}}{1-i} = \frac{1-i(i^2)^{50}}{1-i} = \frac{1-i}{1-i} = 1.$$

3.2. Let ζ_0, ζ_1, ζ_2, ζ_3 and ζ_4 be all the fifth roots of unity. Then for $k = 0, 1, 2, 3, 4$,

$$\zeta_k = e^{i(0+2k\pi)/5} = e^{2k\pi i/5}.$$

3.3. We have

$$\left(\frac{z+1}{z}\right)^5 = 1,$$

which is the same as

$$\left(1+\frac{1}{z}\right)^5 = 1.$$

Let $w = 1 + \frac{1}{z}$. Then we have

$$w_k = e^{2k\pi i/5}, \quad k = 0, 1, 2, 3, 4.$$

Therefore for $k = 0, 1, 2, 3, 4$,

$$\frac{1}{z_k} = e^{2k\pi i/5} - 1,$$

which is the same as

$$z_k = \frac{1}{e^{2k\pi i/5} - 1}.$$