Answers to Assignment 5

8.1. **(Count this question in 2 parts: $n \geq 3$ and $n = 2$.)**
($n \geq 3$) Let $N > 2$. Then

$$\int_{\mathbb{R}^n} \frac{|N_n(x)|}{(1+|x|)^N} dx = \int_{|x|\leq 1} \frac{|N_n(x)|}{(1+|x|)^N} dx + \int_{|x|\geq 1} \frac{|N_n(x)|}{(1+|x|)^N} dx.$$

We first note that

$$\begin{aligned}
\int_{|x|\leq 1} \frac{|N_n(x)|}{(1+|x|)^N} dx &= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{|x|\leq 1} \frac{|x|^{2-n}}{(1+|x|)^N} dx \\
&\leq \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{|x|\leq 1} |x|^{2-n} dx \\
&\leq \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_0^1 \int_{\mathbb{S}^{n-1}} r^{2-n} r^{n-1} dr\, d\sigma \\
&= \frac{1}{n-2} \int_0^1 r dr \\
&= \frac{1}{n-2} \left(\frac{r^2}{2} \right) \Big|_0^1 \\
&= \frac{1}{2n-4} < \infty.
\end{aligned}$$

Note that N can be any positive integer when $|x| \leq 1$. We also note that

$$\begin{aligned}
\int_{|x|\geq 1} \frac{|N_n(x)|}{(1+|x|)^N} dx &= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{|x|\geq 1} \frac{|x|^{2-n}}{(1+|x|)^N} dx \\
&\leq \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{|x|\geq 1} \frac{1}{|x|^N} dx \\
&= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_1^\infty \int_{\mathbb{S}^{n-1}} r^{-N} r^{n-1} dr\, d\sigma \\
&\leq \frac{1}{n-2} \int_1^\infty r^{-N+n-1} dr
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n-2}\left(-\frac{r^{-N+n}}{N-n}\right)\Bigg|_1^\infty \\
&= \frac{1}{(n-2)(N-n)} < \infty.
\end{aligned}
$$

$(n = 2)$ We again have for every positive integer N,

$$
\int_{\mathbb{R}^2} \frac{|N_2(x)|}{(1+|x|)^N} dx = \int_{|x|\le 1} \frac{|N_2(x)|}{(1+|x|)^N} dx + \int_{|x|\ge 1} \frac{|N_2(x)|}{(1+|x|)^N} dx.
$$

Now,

$$
\begin{aligned}
\int_{|x|\le 1} \frac{|N_2(x)|}{(1+|x|)^N} dx &\le \frac{1}{2\pi} \int_{|x|\le 1} |\ln|x||\ dx \\
&= -\frac{1}{2\pi} \int_0^1 \int_{\mathbb{S}^1} (\ln r) r dr\, d\theta \\
&= -\frac{1}{2\pi} \int_0^1 \int_{\mathbb{S}^1} r \ln r\, dr.
\end{aligned}
$$

Let $u = \ln r$ and $dv = r\, dr$. Then

$$
\begin{aligned}
-\int_0^1 r \ln r\, dr &= -\frac{r^2}{2} \ln r \Bigg|_0^1 + \int_0^1 \frac{r}{2} dr \\
&= \frac{r^2}{4}\Bigg|_0^1 = \frac{1}{4}
\end{aligned}
$$

and hence

$$
\int_{|x|\le 1} \frac{|N_2(x)|}{(1+|x|^N} dx < \infty
$$

for all positive integers N. Finally,

$$
\begin{aligned}
\int_{|x|\ge 1} \frac{|N_2(x)|}{(1+|x|)^N} dx &= \frac{1}{2\pi} \int_{|x|\ge 1} \frac{\ln|x|}{(1+|x|)^N} dx \\
&= \frac{1}{2\pi} \int_1^\infty \int_{\mathbb{S}^1} \frac{\ln r}{(1+r)^N} r\, dr\, d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_1^\infty \frac{\ln r}{(1+r)^N} r\, dr \\
&= \int_1^\infty \frac{\ln r}{r^{N-1}} dr.
\end{aligned}
$$

Since $r \le e^r$ for all r with $r \ge 1$, we have

$$\ln r \le r, \quad r \ge 1.$$

So, for $N > 3$,

$$
\begin{aligned}
\int_1^\infty \frac{\ln r}{r^{N-1}} dr &\le \int_1^\infty r^{-N+2} dr \\
&= -\frac{r^{-N+3}}{N-3}\bigg|_1^\infty = \frac{1}{N-3} < \infty.
\end{aligned}
$$

Therefore

$$\int_{|x|\ge 1} \frac{|N_2(x)|}{(1+|x|)^N} dx < \infty$$

for all positive integers N with $N > 3$.

8.2. Let $f \in \mathcal{S}$. Then

$$(N_n * f)(x) = N_n((\tilde{f})_{-x}) = \int_{\mathbb{R}^n} N_n(y) f(x-y)\, dy, \quad x \in \mathbb{R}^n.$$

Therefore for all $x \in \mathbb{R}^n$,

$$f(x) = (\Delta(N_n * f))(x) = \int_{\mathbb{R}^n} N_n(y)(\Delta f)(x-y)\, dy.$$

But for all $x \in \mathbb{R}^n$,

$$
\begin{aligned}
((\Delta N_n) * f)(x) &= (\Delta N_n)((\tilde{f})_{-x}) \\
&= N_n(\Delta(\tilde{f})_{-x}) \\
&= \int_{\mathbb{R}^n} N_n(y)(\Delta f)(x-y)\, dy.
\end{aligned}
$$

So,

$$(\Delta N_n) * f = f = \delta * f, \quad f \in \mathcal{S}.$$

This proves that

$$\Delta N_n = \delta.$$

8.3. Let $f \in \mathcal{S}$. Let u be a solution of

$$\frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2} = f$$

on $\mathbb{R}^2$. Then

$$\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) u = \frac{\partial f}{\partial x_1} - i\frac{\partial f}{\partial x_2}.$$

So, we have

$$\Delta u = \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}.$$

Therefore

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|((\partial_1 f)(y) - i(\partial_2 f)(y))dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|y|((\partial_1 f)(x-y) - i(\partial_2 f)(x-y))\, dy \end{aligned}$$

for all $x \in \mathbb{R}^2$. To put the solution u in the perspective of a Green function or a fundamental solution, we first note that for $j = 1, 2$,

$$(\partial_j N_2)(y) = \frac{1}{2\pi}\frac{1}{|y|}\left(-\frac{1}{2}\frac{1}{|y|}2y_j\right) = -\frac{1}{2\pi}\frac{y_j}{|y|^2}$$

for all $y \in \mathbb{R}^2 \setminus \{0\}$. Thus,

$$(\partial_1 N_2)(y) - i(\partial_2 N_2)(y) = -\frac{1}{2\pi}\frac{y_1 - iy_2}{|y|^2} = -\frac{1}{y_1 + iy_2}$$

for all $y_1 + iy_2$ with $y_1 + iy_2 \neq 0$. So, for all $x \in \mathbb{R}^2$,

$$\begin{aligned} u(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{y_1 + y_2} f(x-y)\, dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{(x_1 - y_1) + i(x_2 - y_2)} f(y)\, dy. \end{aligned}$$

Therefore a Green function for $\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$ is $\frac{1}{2\pi}\frac{1}{x_1+ix_2}$ with $x_1 + ix_2 \neq 0$.

11.1. Since

$$P_t(x) = \frac{2}{|\mathbb{S}^n|} \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}, \quad x \in \mathbb{R}^n,$$

for all $t > 0$, it follows that $P_t \in L^1(\mathbb{R}^n)$ for all $t > 0$. Let N be any positive integer. Then

$$\int_{\mathbb{R}^n} \frac{|P_t(x)|}{(1+|x|)^N} dx \leq \int_{\mathbb{R}^n} |P_t(x)|\, dx < \infty$$

for all $t > 0$. So, P_t is a tempered function on $\mathbb{R}^n$.

11.2. Let u be the function on $\mathbb{R}^n \times (0, \infty)$ given by

$$u(x,t) = P_t(x), \quad x \in \mathbb{R}^n, t > 0.$$

That u is a solution of the PDE on $\mathbb{R}^n \times (0,\infty)$ and

$$u(\cdot, 0) = P_0(\cdot) = \delta$$

are laid out on page 84 of the book.

11.3. The answer is no because u on $\mathbb{R}^n \times [0, \infty)$ defined by

$$u(x,t) = P_t(x) + t, \quad x \in \mathbb{R}^n, t \in [0,\infty),$$

is also a solution.