**Mathematics 2015 3.0 BF**
**Solutions to Test 2**

1. Using polar coordinates,

$$\begin{aligned}
\iint\limits_R e^{-(x^2+y^2)/2} dA &= \int_0^{\pi/2} \int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \\
&= \int_0^{\pi/2} -e^{-r^2/2}\Big|_0^{\infty} d\theta \\
&= \int_0^{\pi/2} d\theta \\
&= \frac{\pi}{2}.
\end{aligned}$$

2.

Let $V$ be the volume. Then

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{2} \int_0^{\sqrt{16-r^2}} r dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{2} \left( z\Big|_0^{\sqrt{16-r^2}} \right) r dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{2} \sqrt{16-r^2} r dr\, d\theta \\
&= \int_0^{2\pi} \left( -\frac{1}{3}(16-r^2)^{3/2}\Big|_0^2 \right) \\
&= \int_0^{2\pi} \frac{1}{3}(16^{3/2} - 12^{3/2})\, d\theta \\
&= \frac{2}{3}\pi(16^{3/2} - 12^{3/2}).
\end{aligned}$$

3. The volume $V$ is

$$\iiint_{E_9} (x^2 + y^2) dV - \iiint_{E_4} (x^2 + y^2) dV = \iiint_{E} x^2 + y^2) dV,$$

where $4 \leq x^2 + y^2 + z^2 \leq 9$. Using spherical coordinates,

$$\begin{aligned} x^2 + y^2 &= \rho^2 \sin^2 \phi \, \cos^2 \theta + \rho^2 \sin^2 \phi \, \sin^2 \theta \\ &= \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) \\ &= \rho^2 \sin^2 \phi, \end{aligned}$$

and therefore

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi} \int_2^3 \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \left. \frac{\rho^5}{5} \right|_2^3 \sin^3 \phi \, d\phi \, d\theta \\ &= \frac{3^5 - 2^5}{5} 2\pi \int_0^{\pi} \sin^3 \phi \, d\phi \\ &= 2\pi \frac{3^5 - 2^5}{5} \int_0^{\pi} (1 - \cos^2 \phi) \sin \, \phi \, d\phi \\ &= 2\pi \frac{3^5 - 2^5}{5} \left. \left( - \cos \, \phi + \frac{\cos^3 \phi}{3} \right) \right|_0^{\pi} \\ &= 2\pi \frac{3^5 - 2^5}{5} \frac{4}{3} \\ &= \frac{8}{15} (3^5 - 2^5) \pi. \end{aligned}$$

4.

We use the change of variables

$$\begin{cases} u = y - x, \\ v = y + x. \end{cases}$$

Thus,

$$\begin{cases} x = \frac{1}{2}v - \frac{1}{2}u, \\ y = \frac{1}{2}v + \frac{1}{2}u. \end{cases}$$

Then

$$\begin{aligned}
&\iint\limits_R \cos\left(\frac{y-x}{y+x}\right) dA \\
= &\iint\limits_S \cos\left(\frac{u}{v}\right) \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}\right| dS \\
= &\int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) \left|\det\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right| du\, dv \\
= &\frac{1}{2}\int_1^2 v \sin\left(\frac{u}{v}\right)\Big|_{-v}^{v} dv \\
= &\int_1^2 v \sin(1)\, dv \\
= &\frac{3}{2}\sin(1).
\end{aligned}$$