Iterative Properties of Pseudo-Differential Operators on Edge Spaces*

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Abstract

Pseudo-differential operators with twisted symbolic estimates play a large role in the calculus on manifolds with edge singularities. We study here aspects of the underlying abstract concept and establish a new result on iteration of quantizations.

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Introduction

The cone, edge and corner pseudo-differential theories of [6], [20], [25], [27], organized as algebras of operators with symbolic structures, suggest an iterative approach. This paper is devoted to new elements of this program. In Sections 1 and 2 we give the abstract edge spaces, twisted symbolic estimates and associated operators. Edge spaces modelled on Hilbert and more general spaces with group action have been introduced in [24]. The edge pseudo-differential calculus in such spaces based on operators with operator-valued symbols arises in the analysis of boundary value problems. See, in particular, [1] for operators with transmission property on the boundaty, and [21] for the case without the transmission property.

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The corresponding boundary symbolic calculus in L^2 spaces on the half-axis and relations to Mellin operators have been studied in Eskin's book [7]. This is one of the sources of the pseudo-differential calculus on manifolds with conical singularities in the sense of [20]. Other results can be traced back to Kondratyev [14], Rabinovich [19], Kondratyev and Oleynik [15]. Operator-valued symbolic structures have also been used in Vishik and Grushin [33] in the context of boundary value problems for certain degenerate operators. The paper [18] of Luke contains an index theorem for elliptic operators of order zero based on operator-valued symbols on L^2 spaces. The lack of the higher order case is due to the lack of spaces with group actions, which are given later in [24]. The development of the singular analysis up to iterative concepts for higher singularities has been outlined in Chapter 10 of [11]. See also [2, 10, 22, 23, 27, 31, 28]. Many specific contributions and applications are given in [3, 4, 5, 8, 9, 29, 32]. Let us finally mention the useful paper [12] where interpolation properties of edge Sobolev spaces have been studied by a more abstract integral transform than the Fourier transform. In Section 3 we establish a new theorem for iterated pseudo-differential operators on edge spaces. This is a result in the larger program of completing the calculus of k-fold iterated corner pseudo-differential operators for $k \geq 2$.

1 Abstract edge spaces

Abstract edge spaces, to be defined in Definition 1.1, play a large role in the following investigations. Spaces of that kind have been introduced and widely investigated in a first version of the edge algebra in [24]. In a more "concrete" form they have already appeared in [21]. In a paper of Hirschmann [12] these spaces are investigated in connection with interpolations and other useful functional analytic properties. Vector-valued spaces without group action in the reference spaces H are employed in Luke [18] in connection with the index theory of elliptic operators with operator-valued symbols. In the paper [33] of Vishik and Grushin, degenerate operators in terms of operator-valued symbols are studied. Certain versions of edge spaces have been applied by Dreher and Witt [5] to hyperbolic problems. In Flad and Harutyunyan [8] the edge algebra machinery, including edge spaces, has been applied to models of particle physics.
Let us first recall some notation and definitions on pseudo-differential operators with operator-valued symbols. The starting point is a Hilbert space H with a group of isomorphisms $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$,

$$\kappa_\delta : H \to H,$$

such that $\delta \to \kappa_\delta h$ defines an element of $C(\mathbb{R}_+, H)$ for every $h \in H$. In that case we say that H is endowed with a group action. There are then constants $c, g > 0$ such that

$$\|\kappa_\delta\|_{\mathcal{L}(H)} \le c(\max\{\delta, \delta^{-1}\})^g. \tag{1.1}$$

More generally, if E is a Fréchet space, written as a projective limit $E = \varprojlim_{j\in\mathbb{N}} E^j$ of Hilbert spaces E^j, continuously embedded in E^0 for all j, a group action on E is defined by a group action on E^0 such that the restriction $\kappa_\delta|_{E^j}$ is a group action on E^j for every j.

Definition 1.1. (i) Let H be a Hilbert space with group action κ. Then for $s \in \mathbb{R}$, $\mathcal{W}^s(\mathbb{R}^q, H)$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm $\| \, \|_{\mathcal{W}^s(\mathbb{R}^q,H)}$ given by

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} = \left\{ \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \, \|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H đ\eta \right\}^{1/2}, \tag{1.2}$$

where $\hat{u}(\eta) = (F_{y\to\eta}u)(\eta)$ is the Fourier transform, $đ\eta := (2\pi)^{-q} d\eta$.
(ii) For a Fréchet space E with group action in the above-mentioned sense, we define $\mathcal{W}^s(\mathbb{R}^q, E)$ by

$$\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j).$$

If necessary, in order to indicate the dependence of the spaces on κ we also write

$$\mathcal{W}^s(\mathbb{R}^q, H)_\kappa \quad \text{and} \quad \mathcal{W}^s(\mathbb{R}^q, E)_\kappa,$$

respectively. Clearly the case id consisting of $\kappa_\delta = \text{id}$ for all $\delta \in \mathbb{R}_+$ is admitted. Then we have

$$\mathcal{W}^s(\mathbb{R}^q, H)_{\text{id}} = H^s(\mathbb{R}^q, H).$$

The spaces in Definition 1.1 are also referred to as abstract edge Sobolev spaces. Recall that the operator K given by

$$Ku = F^{-1}\kappa_{\langle\eta\rangle}Fu$$

induces an isomorphism

$$K : \mathcal{W}^s(\mathbb{R}^q, H)_{\text{id}} \to \mathcal{W}^s(\mathbb{R}^q, H)_\kappa$$

for every $s \in \mathbb{R}$.
For any positive function $w(\eta) \in C(\mathbb{R}^q)$ such that there exist positive constants c_1 and c_2 for which

$$c_1 w(\eta) \le \langle\eta\rangle \le c_2 w(\eta)$$

for all $\eta \in \mathbb{R}^q$, the integral

$$\left\{ \int_{\mathbb{R}^q} w(\eta)^{2s} \|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H đ\eta \right\}^{1/2}$$

gives an equivalent norm to (1.2). The case when

$$w(\eta) = \langle \sigma\eta \rangle ,$$

where $\sigma \in \mathbb{R}_+$ is fixed, is of particular interest to us. Moreover, there are different choices of group actions $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ and $\vartheta = \{\vartheta_\delta\}_{\delta\in\mathbb{R}_+}$ on H such that (1.2) is equivalent to

$$\left\{ \int_{\mathbb{R}^q} \langle \eta \rangle^{2s} \|\vartheta^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|^2_H đ\eta \right\}^{1/2}.$$

This is the case, for instance, for $\vartheta_\delta u = \kappa_{\sigma\delta} u$ for any fixed $\sigma \in \mathbb{R}_+$. More generally, κ is equivalent to ϑ if $\sup_{\delta\in\mathbb{R}_+} \|\kappa_\delta \vartheta_\delta^{-1}\|_{\mathcal{L}(H)} < \infty$. Moreover,

$$\left\{ \int_{\mathbb{R}^q} \langle \sigma\eta \rangle^{2s} \|\kappa^{-1}_{\langle\sigma\eta\rangle} \hat{u}(\eta)\|^2_H đ\eta \right\}^{1/2} \tag{1.3}$$

is equivalent to (1.2) for any fixed $\sigma \in \mathbb{R}_+$. For a fixed $u \in \mathcal{W}^s(\mathbb{R}^q, H)$ we can rewrite (1.3) as

$$\left\{ \int_{\mathbb{R}^q} \langle \eta \rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \hat{u}_\sigma(\eta)\|^2_H đ\eta \right\}^{1/2}$$

for a continuous family $\mathbb{R}_+ \to \mathcal{W}^s(\mathbb{R}^q, H), \sigma \to u_\sigma$. In fact, we may set

$$\hat{u}_\sigma(\eta) = \frac{\langle\sigma\eta\rangle^s}{\langle\eta\rangle^s} \kappa^{-1}_{\langle\sigma\eta\rangle/\langle\eta\rangle} \hat{u}(\eta) \tag{1.4}$$

for all $\eta \in \mathbb{R}^q$.

Theorem 1.2. *Let H be a Hilbert space with group action κ. Then $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ is a Hilbert space with group action $\chi = \{\chi_\delta\}_{\delta\in\mathbb{R}_+}$ defined by*

$$(\chi_\delta u)(y) = \delta^{q/2} (\kappa_\delta u)(\delta y),\ \delta \in \mathbb{R}_+, \tag{1.5}$$

where κ_δ acts pointwise on the values of u in the space H, and for every $p \in \mathbb{N}$ we have

$$\mathcal{W}^s(\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)_\kappa)_\chi = \mathcal{W}^s(\mathbb{R}^{p+q}, H)_\kappa, \tag{1.6}$$

for all $s \in \mathbb{R}$. A similar result holds for a Fréchet space E with group action κ.

Proof. Let us verify that (1.5) defines a group action on $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$. We employ Definition 1.1, assume that $u \in \mathcal{W}^s(\mathbb{R}^q, H)_\kappa$, and compute the $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$-norm of $\chi_\delta u$, $\delta \in \mathbb{R}_+$. Indeed,

$$\begin{aligned}
\|\chi_\delta u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)} &= \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} (\chi_\delta u)^\wedge(\eta)\|^2_H đ\eta = \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \delta^{-q/2} \hat{u}(\delta^{-1}\eta)\|^2_H đ\eta \\
&= \int_{\mathbb{R}^q} \langle\delta\tilde{\eta}\rangle^{2s} \|\kappa^{-1}_{\langle\delta\tilde{\eta}\rangle} \delta^{-q/2} \hat{u}(\tilde{\eta})\|^2_H \delta^q đ\tilde{\eta} = \int_{\mathbb{R}^q} \langle\delta\eta\rangle^{2s} \|\kappa^{-1}_{\langle\delta\eta\rangle} \hat{u}(\eta)\|^2_H đ\eta \\
&= \int_{\mathbb{R}^q} (\frac{\langle\delta\eta\rangle}{\langle\eta\rangle})^{2s} \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\delta\eta\rangle} \hat{u}(\eta)\|^2_H đ\eta,
\end{aligned} \tag{1.7}$$

using the relation

$$F_{y\to\eta}(\vartheta_\delta^{-1}h)(\eta) = \vartheta_\delta(F_{y\to\eta}h)(\eta),\ \delta \in \mathbb{R}_+, \tag{1.8}$$

on a function $h(y), y \in \mathbb{R}^q$, where $(\vartheta_\delta^{-1}h)(y) = \delta^{q/2}h(\delta^{-1}y)$. The right hand side of (1.7) can be written as

$$\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle}^{-1}\hat{u}_\delta(\eta)\|_H^2 đ\eta$$

for $\hat{u}_\delta(\eta) = (\langle\delta\eta\rangle / \langle\eta\rangle)^s \kappa^{-1}_{\langle\delta\eta\rangle/\langle\eta\rangle}\hat{u}(\eta)$ in view of the formula (1.4). As noted before the correspondence $\delta \mapsto F^{-1}_{\eta\to y}\hat{u}_\delta(\eta)$ for fixed $u \in \mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ represents a continuous function on $\delta \in \mathbb{R}_+$ with values in $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$.
For convenience, norms will be identified when they are equivalent. First we write

$$\|f\|^2_{\mathcal{W}^s(\mathbb{R}^{p+q},H)} = \int_{\mathbb{R}^q}\int_{\mathbb{R}^p} (\langle\xi\rangle^2 + |\eta|^2)^s \|\kappa^{-1}_{(\langle\xi\rangle^2+|\eta|^2)^{1/2}}\hat{f}(\xi,\eta)\|^2_H d\xi d\eta,$$

where the "hat" indicates the Fourier transform $F_{(x,y)\to(\xi,\eta)}$. Moreover, employing the expression (1.2) where "hat" has the meaning of $F_{y\to\eta}$, we obtain

$$\begin{aligned}\|f\|^2_{\mathcal{W}^s(\mathbb{R}^{p+q},H)} &= \int_{\mathbb{R}^p}\int_{\mathbb{R}^q} \langle\xi\rangle^{2s}\left(1+\frac{|\eta|^2}{\langle\xi\rangle^2}\right)^s \|\kappa^{-1}_{(\langle\xi\rangle^2+|\eta|^2)^{1/2}}\hat{f}(\xi,\eta)\|^2_H d\eta d\xi \\ &= \int_{\mathbb{R}^p} \langle\xi\rangle^{2s}\left\{\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa^{-1}_{(\langle\xi\rangle^2+\langle\xi\rangle^2|\eta|^2)^{1/2}}\hat{f}(\xi,\langle\xi\rangle\eta)\|^2_H \langle\xi\rangle^q d\eta\right\}d\xi \end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\|f\|^2_{\mathcal{W}^s(\mathbb{R}^p,\mathcal{W}^s(\mathbb{R}^q,H))} &= \int_{\mathbb{R}^p} \langle\xi\rangle^{2s} \|\chi^{-1}_{\langle\xi\rangle}(F_{x\to\xi}f)(\xi,y)\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)}d\xi \\ &= \int_{\mathbb{R}^p} \langle\xi\rangle^{2s} \|\kappa^{-1}_{\langle\xi\rangle}\langle\xi\rangle^{-q/2}(F_{x\to\xi}f)(\xi,\langle\xi\rangle^{-1}y)\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)}d\xi \\ &= \int_{\mathbb{R}^p} \langle\xi\rangle^{2s}\left\{\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\xi\rangle}\kappa^{-1}_{\langle\eta\rangle}\langle\xi\rangle^{q/2}\hat{f}(\xi,\langle\xi\rangle\eta)\|^2_H d\eta\right\}d\xi. \end{aligned} \tag{1.10}$$

In the identification of the expressions in $\{\cdots\}$ occurring in (1.9) and (1.10) we have employed the relation (1.8) and the fact that $\kappa^{-1}_{\langle\xi\rangle}\kappa^{-1}_{\langle\eta\rangle} = \kappa^{-1}_{\langle\xi\rangle\langle\eta\rangle}$. □

The equation (1.6) is an extension of Lemma 1 in Subsection 3.1.1 of [25], i.e., for the case $H = \mathbb{C}$ and $\kappa_\delta = \mathrm{id}_{\mathbb{C}}, \delta \in \mathbb{R}_+$, i.e.,

$$\mathcal{W}^s(\mathbb{R}^p, H^s(\mathbb{R}^q)) = H^s(\mathbb{R}^{p+q})$$

when $H^s(\mathbb{R}^q)$ is endowed with the group action $(\kappa_\delta u)(y) = \delta^{q/2}u(\delta y), \delta \in \mathbb{R}_+$. The equation (1.6) is formula (24) in Subsection 3.1.2 of [25]. Details are given in Proposition 1.3.44 of [26], however, under the some extra assumptions as in [25]. The assumptions are, in fact, redundant. We have presented the general proof here since (1.6) belongs to the iterative concept of corner pseudo-differential operators.

2 Symbols with twisted estimates

Given two Hilbert spaces H and $\widetilde{H}$ with group actions κ and $\tilde{\kappa}$, respectively, a function $f \in C^\infty(\mathbb{R}^q_\eta \setminus \{0\}, \mathcal{L}(H, \widetilde{H}))$ is called twisted homogeneous in $\eta \in \mathbb{R}^q \setminus \{0\}$ of order $\mu \in \mathbb{R}$ if

$$f(\delta\eta) = \delta^\mu \tilde{\kappa}_\delta f(\eta) \kappa_\delta^{-1}$$

for all $\delta \in \mathbb{R}_+$. Let

$$S^{(\mu)}(\mathbb{R}^q \setminus \{0\}; H, \widetilde{H}) \tag{2.1}$$

denote the space of those functions f. Then, if $\chi(\eta)$ is an excision function on $\mathbb{R}^q$, i.e., $\chi \in C^\infty(\mathbb{R}^q), \chi(\eta) = 0$ for $|\eta| < \varepsilon_0, \chi(\eta) = 1$ for $|\eta| > \varepsilon_1$ for some $0 < \varepsilon_0 < \varepsilon_1$, the function $a(\eta) := \chi(\eta) f(\eta)$ is an example of an operator-valued symbol in the following sense. The space

$$S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H}),$$

$\Omega \subseteq \mathbb{R}^p$ open, is defined to be the set of all $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ such that

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle} \{D^\alpha_y D^\beta_\eta a(y, \eta)\} \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H, \widetilde{H})} \leq c \langle\eta\rangle^{\mu - |\beta|} \tag{2.2}$$

for all $(y, \eta) \in K \times \mathbb{R}^q, K \Subset \Omega, \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$, for some constant $c = c(\alpha, \beta, K) > 0$. The subspace $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ of classical symbols is defined as the set of all $a(y, \eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ such that there are functions $a_{(\mu-j)}(y, \eta) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H})$, $j \in \mathbb{N}$, with

$$r_{N+1}(y, \eta) := a(y, \eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y, \eta) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^q; H, \widetilde{H}) \tag{2.3}$$

for every $N \in \mathbb{N}$.

Example 2.1. Let $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ and

$$a(y, \delta\eta) = \delta^\mu \tilde{\kappa}_\delta a(y, \eta) \kappa_\delta^{-1}$$

for all $\delta \geq 1, |\eta| \geq c$ for some $c > 0$. Then $a(y, \eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$.

From the definition it follows that $S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ is a Fréchet space where the semi-norms are the best constants c in the estimates (2.2). Also $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ is Fréchet in the projective limit topology of the mappings $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H}) \to S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H}), a(y, \eta) \to a_{(\mu-j)}(y, \eta), j \in \mathbb{N}$, and $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H}) \to S^{\mu-(N+1)}(\Omega \times \mathbb{R}^q; H, \widetilde{H}), a(y, \eta) \to r_{N+1}(y, \eta)$, cf. (2.3).
If a consideration is valid in the classical or the general case we write subscript "(cl)".

By

$$S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H}) \tag{2.4}$$

we denote the subspaces of symbols with constant coefficients, i.e., which are independent of y. The space (2.4) is closed in $S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$. Then

$$S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H}) = C^{\infty}(\Omega, S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H})) = C^{\infty}(\Omega)\hat{\otimes}_{\pi} S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H}), \tag{2.5}$$

where $\hat{\otimes}_{\pi}$ denotes the (completed) projective tensor product between the involved spaces.

Clearly our spaces of symbols depend on the choice of $\kappa, \tilde{\kappa}$. If necessary, then we write

$$S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})_{\kappa,\tilde{\kappa}} \tag{2.6}$$

instead of $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$.

Lemma 2.2. *Let H and $\widetilde{H}$ be Hilbert spaces with group actions κ and $\tilde{\kappa}$, respectively, and let $T : H \to L$ and $\widetilde{T} : \widetilde{H} \to \widetilde{L}$ be isomorphisms to Hilbert spaces L and $\widetilde{L}$, respectively. Then the spaces $L, \widetilde{L}$ are endowed with group actions λ_{δ}, $\tilde{\lambda}_{\delta}$ given by*

$$\lambda_{\delta} = T\kappa_{\delta}T^{-1}$$

and

$$\tilde{\lambda}_{\delta} = \widetilde{T}\tilde{\kappa}_{\delta}\widetilde{T}^{-1},$$

and $T, \widetilde{T}$ induce an isomorphism

$$Q : S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})_{\kappa,\tilde{\kappa}} \to S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; L, \widetilde{L})_{\lambda,\tilde{\lambda}},$$

where

$$a(y,\eta) \mapsto b(y,\eta)$$

and

$$b(y,\eta) = \widetilde{T} \circ a(y,\eta) \circ T^{-1}, \mu \in \mathbb{R}.$$

Proof. First note that

$$b(y,\eta) = \widetilde{T} \circ a(y,\eta) \circ T^{-1} \in C^{\infty}(\Omega \times \mathbb{R}^q, \mathcal{L}(L, \widetilde{L})). \tag{2.7}$$

Moreover, we have

$$D^{\alpha}_{y} D^{\beta}_{\eta} b(y,\eta) = \widetilde{T} D^{\alpha}_{y} D^{\beta}_{\eta} a(y,\eta) T^{-1}$$

for arbitrary $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$. Let us now check that $b(y,\eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; L, \widetilde{L})_{\lambda,\tilde{\lambda}}$. To this end we have to verify the symbolic estimates to the effect that

$$\|\tilde{\lambda}^{-1}_{\langle\eta\rangle} \{D^{\alpha}_{y} D^{\beta}_{\eta} b(y,\eta)\} \lambda_{\langle\eta\rangle}\|_{\mathcal{L}(L,\widetilde{L})} \leq c \langle\eta\rangle^{\mu-|\beta|}$$

for all $(y, \eta) \in K \times \mathbb{R}^q, K \Subset \Omega$, and $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q, c = c(\alpha, \beta, K) > 0$. From (2.2) and (2.7) we obtain

$$\begin{aligned} & \|\widetilde{T}^{-1}\tilde{\lambda}^{-1}_{\langle\eta\rangle}\widetilde{T}\{D^\alpha_y D^\beta_\eta \widetilde{T}^{-1} b(y,\eta)T\}T^{-1}\lambda_{\langle\eta\rangle}T\|_{\mathcal{L}(H,\widetilde{H})} \\ = \; & \|\widetilde{T}^{-1}\tilde{\lambda}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta b(y,\eta)\}\lambda_{\langle\eta\rangle}T\|_{\mathcal{L}(H,\widetilde{H})} \\ = \; & \|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq c\,\langle\eta\rangle^{\mu-|\beta|}, \end{aligned}$$

using the fact that the operator

$$Q : C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(L, \widetilde{L})) \to C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$$

defined by

$$Q : b(y,\eta) \mapsto \widetilde{T}^{-1} b(y,\eta) T$$

has the property that

$$D^\alpha_y D^\beta_\eta (Qb)(y,\eta) = Q(D^\alpha_y D^\beta_\eta b)(y,\eta),$$

or, equivalently,

$$\widetilde{T}\{D^\alpha_y D^\beta_\eta \widetilde{T}^{-1} b(y,\eta)T\}T^{-1} = D^\alpha_y D^\beta_\eta b(y,\eta).$$

The assertion for classical symbols is a consequence of the fact that $\widetilde{T} \circ a_{(\mu-j)} \circ T^{-1}$ induces isomorphisms

$$S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H})_{\kappa,\tilde{\kappa}} \to S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); L, \widetilde{L})_{\lambda,\tilde{\lambda}}$$

for every j; here subscripts $\kappa, \tilde{\kappa}; \lambda, \tilde{\lambda}$ have a similar meaning as in (2.6). □

Let Ω be an open subset of $\mathbb{R}^q$. Then for $(y, y') \in \Omega \times \Omega$, and $a(y, y', \eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; H, \widetilde{H})$, we define $\mathrm{Op}_y(a)u(y)$ by

$$\mathrm{Op}_y(a)u(y) = \int_{\mathbb{R}^q}\int_{\mathbb{R}^q} e^{i(y-y')\eta} a(y, y', \eta) u(y') dy' đ\eta,$$

first for $u \in C^\infty_0(\Omega, H)$, and define $L^\mu_{(\mathrm{cl})}(\Omega; H, \tilde{H})$ by

$$L^\mu_{(\mathrm{cl})}(\Omega; H, \widetilde{H}) = \{\mathrm{Op}_y(a) : a(y, y', \eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; H, \widetilde{H})\},$$

and define $L^{-\infty}(\Omega; H, \tilde{H})$ by

$$L^{-\infty}(\Omega; H, \widetilde{H}) = \bigcap_{\mu \in \mathbb{R}} L^\mu(\Omega; H, \widetilde{H}).$$

Recall from [25] or [26] that for $H = \widetilde{H} = \mathbb{C}$ and the trivial group action id both on H and $\widetilde{H}$, we recover the spaces $S^{\mu}_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q)$ of scalar symbols and pseudo-differential operators $L^{\mu}_{(\mathrm{cl})}(\Omega)$. In particular, $L^{-\infty}(\Omega) = \bigcap_{\mu\in\mathbb{R}} L^{\mu}(\Omega)$ coincides with the space of smoothing operators, i.e., operators of the form

$$C_0^{\infty}(\Omega) \ni u \mapsto \int_{\Omega} c(y, y')u(y')dy'$$

for some $c(y, y') \in C^{\infty}(\Omega \times \Omega)$.
There are also variants with parameter $\lambda \in \mathbb{R}^l$, namely,

$$S^{\mu}_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{q+l}_{\eta,\lambda}; H, \widetilde{H}) \quad \text{and} \quad L^{\mu}_{(\mathrm{cl})}(\Omega; H, \widetilde{H}; \mathbb{R}^l_{\lambda}).$$

In this case we define $L^{-\infty}(\Omega; H.\tilde{H}; \mathbb{R}^l)$ by

$$L^{-\infty}(\Omega; H, \widetilde{H}; \mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(\Omega; H, \widetilde{H})). \tag{2.8}$$

An $A \in L^{\mu}(\Omega; H, \widetilde{H})$ induces a continuous operator

$$A : C_0^{\infty}(\Omega, H) \to C^{\infty}(\Omega, H). \tag{2.9}$$

Proposition 2.3. [26, Proposition 1.3.24] *Let* $a(\eta) \in S^{\mu}(\mathbb{R}^q_{\eta}; H, \widetilde{H})$*. Then* $\mathrm{Op}(a)$ *extends to a continuous operator*

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H})$$

for every $s \in \mathbb{R}$*.*

The proof in this case is straightforward. In fact, by virtue of $\mathrm{Op}(a) = F^{-1}aF$ for the Fourier transform F in $y \in \mathbb{R}^q$ we have

$$\begin{aligned} \|\mathrm{Op}(a)u\|^2_{\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H})} &= \int_{\mathbb{R}^q} \langle\eta\rangle^{2(s-\mu)} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\eta)\hat{u}(\eta)\|^2_{\widetilde{H}} đ\eta \\ &\leq \int_{\mathbb{R}^q} \langle\eta\rangle^{2(s-\mu)} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\eta)\kappa_{\langle\eta\rangle}\|^2_{\mathcal{L}(H,\widetilde{H})} \|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H đ\eta \leq c^2\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)} \end{aligned}$$

where $c = \sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})}$, which is finite because of (2.2). In particular, it follows that $a \mapsto \mathrm{Op}(a)$ induces a continuous operator

$$S^{\mu}(\mathbb{R}^q; H, \widetilde{H}) \to \mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H})) \tag{2.10}$$

for every $s \in \mathbb{R}$.
More generally, for $a(y, \eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ we have for all $s \in \mathbb{R}$, continuous operators

$$\mathrm{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, H) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{H}) \tag{2.11}$$

between comp/loc-versions of the abstract edge Sobolev spaces over an open set $\Omega \subseteq \mathbb{R}^q$.
There are different ways of proving such a continuity. One relatively "unspecific" way is the tensor product argument. This refers to the fact that when E and F are Fréchet spaces, every element g in the completed projective tensor product $E\hat{\otimes}_\pi F$ can be written as a convergent sum $\sum_{j=0}^{\infty} \nu_j e_j \otimes f_j$ for suitable $\nu_j \in \mathbb{C}, \sum_{j=0}^{\infty} |\nu_j| < \infty$, and $e_j \in E, f_j \in F, j \in \mathbb{N}$, tending to zero in the respective spaces, as $j \to \infty$. In order to show the continuity of (2.11), we employ the fact that the operator $\mathcal{M}_\varphi$ of multiplication by $\varphi \in C_0^\infty(\Omega)$ induces a continuous operator

$$\mathcal{M}_\varphi : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^s(\mathbb{R}^q, H),$$

where

$$\|\mathcal{M}_\varphi\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^s(\mathbb{R}^q,H))} \to 0 \text{ as } \varphi \to 0 \text{ in } C_0^\infty(\Omega), \tag{2.12}$$

see [26, Proposition 1.3.34]. Because of (2.5), we can represent $a(y, \eta)$ as a convergent sum

$$a(y, \eta) = \sum_{j=0}^{\infty} \nu_j \mathcal{M}_{\varphi_j} a_j(\eta)$$

for sequences $\varphi_j \in C^\infty(\Omega), a_j \in S^\mu(\mathbb{R}^q; H, \widetilde{H})$ tending to zero in the respective spaces as $j \to \infty$, and $\nu_j \in \mathbb{C}, \sum_{j=0}^{\infty} |\nu_j| < \infty$. Then (2.11) follows from (2.10) and (2.12), using the fact that

$$\mathrm{Op}(a) = \sum_{j=0}^{\infty} \nu_j \mathcal{M}_{\varphi_j} \mathrm{Op}(a_j)$$

converges in $\mathcal{L}([\psi]\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{H}))$ for every $\psi \in C_0^\infty(\Omega)$.
Here if E is a Fréchet space that is a module over an algebra, then for any ψ in that algebra we define

$$[\psi]E = \text{completion of } \{\psi e : e \in E\} \text{ in } E. \tag{2.13}$$

Similar arguments of proving the continuity of $\mathrm{Op}(a)$ between edge spaces can be found in [25, Subsection 3.2.1, Theorem 6], however, under an extra assumption that turns out to be superfluous. Other methods are applied in [26, Theorem 1.3.59] or in [32].

Remark 2.4. Let $a_0 \in \mathcal{L}(H, \widetilde{H})$ and assume that

$$\tilde{\kappa}_\delta a_0 \kappa_\delta^{-1} \in C^\infty(\mathbb{R}_+, \mathcal{L}(H, \widetilde{H})). \tag{2.14}$$

Then $a(\eta)$ defined by

$$a(\eta) = \langle \eta \rangle^\mu \, \tilde{\kappa}_{\langle \eta \rangle} a_0 \kappa_{\langle \eta \rangle}^{-1}$$

is in $S^\mu_{\mathrm{cl}}(\mathbb{R}^q; H, \widetilde{H})$.

It also makes sense to consider $g(\eta)$ given by

$$g(\eta) = [\eta]^{\mu} \tilde{\kappa}_{[\eta]} a_0 \kappa_{[\eta]}^{-1}$$

for any fixed strictly positive function $[\cdot]$ in $C^{\infty}(\mathbb{R}^q_{\eta})$ such that $[\eta] = |\eta|$ for $|\eta| \geq c$ for some $c > 0$. Clearly $g(\eta)$ is C^{∞} in η if and only if $a(\eta)$ is C^{∞} in η. For g, we have

$$g(\sigma\eta) = [\sigma\eta]^{\mu} \tilde{\kappa}_{[\sigma\eta]} a_0 \kappa_{[\sigma\eta]}^{-1} = \sigma^{\mu} [\eta]^{\mu} \tilde{\kappa}_{\sigma} \tilde{\kappa}_{[\eta]} a_0 \kappa_{[\eta]}^{-1} \kappa_{\sigma}^{-1} = \sigma^{\mu} \tilde{\kappa}_{\sigma} g(\eta) \kappa_{\sigma}^{-1}$$

for $\sigma \geq 1$ and $|\eta| \geq c$. We also have $g(\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q; H, \widetilde{H})$. See Example 2.1.

Remark 2.5. It can happen that the property (2.14) is violated. For instance, Let $H = \widetilde{H} = L^2(\mathbb{R}^n)$ and let $a_0 = \mathcal{M}_f$ be the operator of multiplication by a function f such that $f \equiv 1$ for $|x| \geq 1$, $f \equiv \frac{1}{2}$ for $|x| < 1$. Then for κ_{δ} defined by

$$\kappa_{\delta} u(x) = u(\delta x), \quad \delta \in \mathbb{R}_+,$$

we have

$$\kappa_{\delta} \mathcal{M}_f \kappa_{\delta}^{-1} u = \mathcal{M}_{f_{\delta}} u$$

for $f_{\delta}(x) = f(\delta x)$. However, because of the discontinuity of f we cannot differentiate f_{δ} with respect to δ.

Note that if we consider an operator $c \in \mathcal{L}(H, \widetilde{H})$ such that $\kappa_{\delta} c \kappa_{\delta}^{-1}$ does not belong to $C^{\infty}(\mathbb{R}_+, \mathcal{L}(H, \widetilde{H}))$, we can generate smoothness by a mollifying process [17, 25]. Indeed, let $c(\delta) = \kappa_{\delta} c \kappa_{\delta}^{-1}$ and let $a(\varrho)$ be defined by

$$a(\varrho) = \int_0^{\infty} \varrho^{-1} \varphi\Big(\frac{\varrho - \delta}{\varrho}\Big) c(\delta) d\delta$$

for a function $\varphi \in C_0^{\infty}(\mathbb{R})$ such that $\operatorname{supp} \varphi \subset [-\varepsilon, \varepsilon]$ for some $0 < \varepsilon < 1/2$ and $\int_{-\infty}^{\infty} \varphi(\delta) d\delta = 1$. Then we have $a(\varrho) \in C^{\infty}(\mathbb{R}_+, \mathcal{L}(H, \widetilde{H}))$. Moreover, for $h \in H$, we have

$$\begin{aligned}
\tilde{\kappa}_{\sigma} a(\varrho) \kappa_{\sigma}^{-1} h &= \int_0^{\infty} \varrho^{-1} \varphi\Big(\frac{\varrho - \delta}{\varrho}\Big) \tilde{\kappa}_{\sigma} c(\delta) \kappa_{\sigma}^{-1} h d\delta = \int_0^{\infty} \varrho^{-1} \varphi\Big(\frac{\varrho - \delta}{\varrho}\Big) c(\sigma\delta) h d\delta \\
&= \int_0^{\infty} \varrho^{-1} \varphi\Big(\frac{\varrho - \frac{\tilde{\delta}}{\sigma}}{\varrho}\Big) c(\tilde{\delta}) h \sigma^{-1} d\tilde{\delta} = \int_0^{\infty} \sigma^{-1} \varrho^{-1} \varphi\Big(\frac{\sigma\varrho - \tilde{\delta}}{\sigma\varrho}\Big) c(\tilde{\delta}) h d\tilde{\delta} \\
&= a(\sigma\varrho).
\end{aligned}$$

In other words, we obtain the homogeneity

$$a(\sigma\varrho) = \kappa_{\sigma} a(\varrho) \kappa_{\sigma}^{-1}, \ \sigma \in \mathbb{R}_+.$$

We conclude that $\tilde{\kappa}_{\langle\eta\rangle} a_1 \kappa_{\langle\eta\rangle}^{-1} \in C^\infty(\mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ and $a(\eta)$ given by

$$a(\eta) = \langle\eta\rangle^\mu \, \tilde{\kappa}_{\langle\eta\rangle} a_1 \kappa_{\langle\eta\rangle}^{-1}$$

is in $S^\mu_{\mathrm{cl}}(\mathbb{R}^q; H, \widetilde{H})$.
Concrete cases in connection with Remark 2.4 are as follows. Let

$$b(x, x', \xi) \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^n_{x'}, S^\mu_{(\mathrm{cl})}(\mathbb{R}^n_\xi))$$

and let $a_0 u = \mathrm{Op}(b)u$, where

$$\mathrm{Op}(b)u = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-x')\xi} b(x, x', \xi) u(x') dx' đ\xi.$$

Then $a_0 : H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n)$ is continuous for every $s \in \mathbb{R}$. Let κ_δ be given by

$$(\kappa_\delta u)(x) = u(\delta x), \quad \delta \in \mathbb{R}_+,$$

and compute $\kappa_\delta a_0 \kappa_\delta^{-1}$. By some obvious computations, we obtain

$$\begin{aligned}(\kappa_\delta \mathrm{Op}(b) \kappa_\delta^{-1} u)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\delta x - x')\xi} b(\delta x, x', \xi) u(\delta^{-1} x') dx' đ\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-\tilde{x})\tilde{\xi}} b(\delta x, \delta\tilde{x}, \delta^{-1}\tilde{\xi}) u(\tilde{x}) d\tilde{x} đ\tilde{\xi}.\end{aligned}$$

Thus

$$\kappa_\delta \mathrm{Op}(b) \kappa_\delta^{-1} = \mathrm{Op}(b_\delta) \tag{2.15}$$

for $b_\delta(x, x', \xi) = b(\delta x, \delta x', \delta^{-1}\xi)$. It follows in this case that

$$D^k_\delta(\kappa_\delta \mathrm{Op}(b) \kappa_\delta^{-1}) = \mathrm{Op}(D^k_\delta b_\delta)$$

for every $k \in \mathbb{N}$. Therefore

$$\kappa_\delta \mathrm{Op}(b) \kappa_\delta^{-1} \in C^\infty(\mathbb{R}_+, \mathcal{L}(H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n))).$$

Applying the construction to symbols $b(x, y, \xi, \eta) \in \mathcal{S}(\mathbb{R}^n_x, S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_y \times \mathbb{R}^n_\xi \times \mathbb{R}^q_\eta))$, it follows that

$$\kappa_{\langle\eta\rangle}^{-1} \mathrm{Op}_x(b)(y, \eta) \kappa_{\langle\eta\rangle} = \mathrm{Op}_x(b_{\mathrm{dec}})(y, \eta),$$

where

$$b_{\mathrm{dec}}(x, y, \xi, \eta) = b(\langle\eta\rangle^{-1} x, y, \langle\eta\rangle \xi, \eta). \tag{2.16}$$

The symbol $b_{\mathrm{dec}}(x, y, \xi, \eta)$ is often useful as a "decoupled" symbol. The decoupling $b \to b_{\mathrm{dec}}$ is known [29, Proposition 2.2.1] to generate a continuous operator

$$S^\mu_{(\mathrm{cl})}(\mathbb{R}^n_x \times \mathbb{R}^q_y \times \mathbb{R}^n_\xi \times \mathbb{R}^q_\eta) \to S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta, S^\mu_{(\mathrm{cl})}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)),$$

Let us also recall that in boundary value problems (when $n = 1$) or in the edge pseudo-differential calculus (for arbitrary n), it is useful to interpret $\mathrm{Op}_y(b_0)(y, \eta)$ for $b_0(y, \xi, \eta) = b(0, y, \xi, \eta)$ as an element of

$$\mathrm{Op}(b_0)(y, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta; H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n)),$$

$s \in \mathbb{R}$, where the symbol space on the right refers to $\kappa_\delta : u(x) \to \delta^{\frac{n}{2}} u(\delta x)$, both on $H^s(\mathbb{R}^n)$ and $H^{s-\mu}(\mathbb{R}^n)$.

3 Operators in iterated representation

Theorem 3.1. *Let H and $\widetilde{H}$ be Hilbert spaces with group actions κ and $\tilde{\kappa}$, respectively. Then for $a(\xi, \eta) \in S^\mu(\mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}}$ and $p(\eta)$ defined by*

$$p(\eta) = \mathrm{Op}_x(a)(\eta),$$

we have

$$p(\eta) \in S^\mu(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}}$$

for every $s \in \mathbb{R}$, where

$$(\chi_\delta f)(x) = \delta^{p/2}(\kappa_\delta f)(\delta x)$$

and

$$(\tilde{\chi}_\delta \tilde{f})(x) = \delta^{p/2}(\tilde{\kappa}_\delta \tilde{f})(\delta x), \delta \in \mathbb{R}_+.$$

Proof. Indeed,

$$\|\tilde{\kappa}^{-1}_{\langle \xi,\eta\rangle}\{D^\alpha_{\xi,\eta} a(\xi,\eta)\}\kappa_{\langle\xi,\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \le c\,\langle \xi,\eta\rangle^{\mu-|\alpha|} \tag{3.1}$$

for every $\alpha \in \mathbb{N}^{p+q}$, $(\xi, \eta) \in \mathbb{R}^{p+q}$, for some $c = c(\alpha) > 0$. We first note that

$$\langle \langle\eta\rangle\,\xi,\eta\rangle^2 = 1 + \langle\eta\rangle^2 |\xi|^2 + |\eta|^2 = \langle\eta\rangle^2 + \langle\eta\rangle^2|\xi|^2 = \langle\xi\rangle^2\langle\eta\rangle^2 . \tag{3.2}$$

In (3.1) we first assume $\alpha = 0$ and obtain

$$\|\tilde{\kappa}^{-1}_{\langle\langle\eta\rangle\xi,\eta\rangle} a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\langle\eta\rangle\xi,\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \le c\,\langle\langle\eta\rangle\,\xi,\eta\rangle^\mu = c\,\langle\xi\rangle^\mu\langle\eta\rangle^\mu . \tag{3.3}$$

For $p(\eta) = \mathrm{Op}_x(a)(\eta)$ and $v_\eta(x) = \tilde{\chi}^{-1}_{\langle\eta\rangle} p(\eta)\chi_{\langle\eta\rangle}u(x)$, we have

$$\begin{aligned}
v_\eta(x) &= \tilde{\chi}^{-1}_{\langle\eta\rangle}\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(x-x')\xi} a(\xi,\eta)\chi_{\langle\eta\rangle}u(x')dx'đ\xi \\
&= \tilde{\kappa}^{-1}_{\langle\eta\rangle}\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(\langle\eta\rangle^{-1}x-x')\xi} a(\xi,\eta)\kappa_{\langle\eta\rangle}u(\langle\eta\rangle\,x')dx'đ\xi \\
&= \tilde{\kappa}^{-1}_{\langle\eta\rangle}\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(x-x')\langle\eta\rangle^{-1}\xi} a(\xi,\eta)\kappa_{\langle\eta\rangle}u(x')\,\langle\eta\rangle^{-p}\,dx'đ\xi \\
&= \tilde{\kappa}^{-1}_{\langle\eta\rangle}\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(x-x')\xi} a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\eta\rangle}u(x')dx'đ\xi \\
&= \tilde{\kappa}^{-1}_{\langle\eta\rangle}F^{-1}_{\xi\to x}a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\eta\rangle}\hat{u}(\xi).
\end{aligned}$$

Thus

$$(\tilde{\chi}^{-1}_{\langle\eta\rangle} p(\eta)\chi_{\langle\eta\rangle} u)^{\wedge}(\xi) = \tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\eta\rangle}\hat{u}(\xi). \tag{3.4}$$

In view of (3.2), the estimate (3.3) gives

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\tilde{\kappa}^{-1}_{\langle\xi\rangle} a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\xi\rangle}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq c\,\langle\xi\rangle^{\mu}\,\langle\eta\rangle^{\mu}\,. \tag{3.5}$$

Let us now show that

$$\|\tilde{\chi}^{-1}_{\langle\eta\rangle} p(\eta)\chi_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{W}^{s}(\mathbb{R}^p,H),\mathcal{W}^{\tilde{s}}(\mathbb{R}^p,\widetilde{H}))} \leq c\,\langle\eta\rangle^{\mu}\,. \tag{3.6}$$

For $u(x) \in \mathcal{W}^{s}(\mathbb{R}^p_x, H)$ and $\tilde{s} = s-\mu$, we have

$$\begin{aligned}
\|v_\eta\|^2_{\mathcal{W}^{\tilde{s}}(\mathbb{R}^p,\widetilde{H})} &= \int_{\mathbb{R}^p} \langle\xi\rangle^{2\tilde{s}}\,\|\tilde{\kappa}^{-1}_{\langle\xi\rangle}(\tilde{\chi}^{-1}_{\langle\eta\rangle} p(\eta)\chi_{\langle\eta\rangle} u)^{\wedge}(\xi)\|^2_{\widetilde{H}} d\xi \\
&= \int_{\mathbb{R}^p} \langle\xi\rangle^{2\tilde{s}}\,\|\tilde{\kappa}^{-1}_{\langle\xi\rangle}\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\eta\rangle}\hat{u}(\xi)\|^2_{\widetilde{H}} d\xi.
\end{aligned} \tag{3.7}$$

Here the relation (3.4) is used. The right hand side of (3.7) is equal to

$$\begin{aligned}
& \int_{\mathbb{R}^p} \langle\xi\rangle^{2(s-\mu)}\,\|\tilde{\kappa}^{-1}_{\langle\xi\rangle}\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\eta\rangle}\kappa_{\langle\xi\rangle}\kappa^{-1}_{\langle\xi\rangle}\hat{u}(\xi)\|^2_{\widetilde{H}} d\xi \\
\leq\ & \int_{\mathbb{R}^p} \langle\xi\rangle^{-2\mu}\,\|\tilde{\kappa}^{-1}_{\langle\xi\rangle}\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\eta\rangle}\kappa_{\langle\xi\rangle}\|^2_{\mathcal{L}(H,\widetilde{H})}\,\langle\xi\rangle^{2s}\,\|\kappa^{-1}_{\langle\xi\rangle}\hat{u}(\xi)\|^2_{H} d\xi \\
\leq\ & c^2\,\langle\eta\rangle^{2\mu}\int_{\mathbb{R}^p} \langle\xi\rangle^{2s}\,\|\kappa^{-1}_{\langle\xi\rangle}\hat{u}(\xi)\|^2_{H} d\xi,
\end{aligned}$$

where

$$c = \sup_{(\xi,\eta)\in\mathbb{R}^{p+q}} \langle\xi\rangle^{-\mu}\,\langle\eta\rangle^{-\mu}\,\|\tilde{\kappa}^{-1}_{\langle\xi\rangle}\tilde{\kappa}^{-1}_{\langle\eta\rangle} a(\langle\eta\rangle\,\xi,\eta)\kappa_{\langle\eta\rangle}\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H,\widetilde{H})}$$

comes from the estimate (3.5). Summing up, we have proved the symbolic estimate (3.6) for every $s \in \mathbb{R}$. Finally, using

$$D^{\beta}_{\eta} p(\eta) = \mathrm{Op}_x(D^{\beta}_{\eta} a)(\eta)$$

and

$$D^{\beta}_{\eta} a(\xi,\eta) \in S^{\mu-|\beta|}(\mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})$$

from (3.6), we immediately obtain

$$\|\tilde{\chi}^{-1}_{\langle\eta\rangle} D^{\beta}_{\eta} p(\eta)\chi_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{W}^{s}(\mathbb{R}^p,H),\mathcal{W}^{s-\mu+|\beta|}(\mathbb{R}^p,\widetilde{H}))} \leq c\,\langle\eta\rangle^{\mu-|\beta|} \tag{3.8}$$

for all $\beta \in \mathbb{N}^q$. This then implies a similar estimate for $\mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})$ on the left hand side of (3.8). So, the proof is complete. □

Proposition 3.2. *If* $a(\xi,\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}}$ *and* $p(\eta) = \mathrm{Op}_x(a)(\eta)$*, then*

$$p(\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q_{\eta}; \mathcal{W}^s(\mathbb{R}^p, H)_{\kappa}, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}}$$

for all $s \in \mathbb{R}$.

Proof. We first look at $a_{(\mu)}(\xi,\eta)$, the twisted homogeneous principal component of $a(\xi,\eta)$. Then for $p_{(\mu)}(\eta) = \mathrm{Op}_x(a_{(\mu)})(\eta)$ with $\eta \neq 0$, we have

$$p_{(\mu)}(\delta\eta) = \delta^{\mu}\tilde{\chi}_{\delta} p_{(\mu)}(\eta)\chi_{\delta}^{-1}$$

for every $\delta \in \mathbb{R}_+$. In fact,

$$\begin{aligned} p_{(\mu)}(\delta\eta)u(x) &= \int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(x-x')\xi} a_{(\mu)}(\xi,\delta\eta)u(x')dx'đ\xi \\ &= \int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(x-x')\xi} a_{(\mu)}(\delta^{-1}\delta\xi,\delta\eta)u(x')dx'đ\xi \\ &= \int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(x-x')\xi}\delta^{\mu}\tilde{\kappa}_{\delta} a_{(\mu)}(\delta^{-1}\xi,\eta)\kappa_{\delta}^{-1}u(x')dx'đ\xi \\ &= \delta^{\mu}\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(x-x')\delta\xi''}\tilde{\kappa}_{\delta} a_{(\mu)}(\xi'',\eta)\kappa_{\delta}^{-1}u(x')\delta^p dx'đ\xi'' \\ &= \delta^{\mu}\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(\delta x-x'')\xi''}\tilde{\kappa}_{\delta} a_{(\mu)}(\xi'',\eta)\kappa_{\delta}^{-1}u(\delta^{-1}x'')dx''đ\xi'' \\ &= \delta^{\mu}\tilde{\chi}_{\delta}\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i(x-x')\xi} a_{(\mu)}(\xi,\eta)\chi_{\delta}^{-1}u(x')dx'đ\xi. \end{aligned}$$

For every $N \in \mathbb{N}$ we can write

$$a(\xi,\eta) = \sum_{j=0}^{N}\chi(\xi,\eta)a_{(\mu-j)}(\xi,\eta) + r_{N+1}(\xi,\eta)$$

for any excision function $\chi(\xi,\eta)$ in $\mathbb{R}^{p+q}$ and some

$$r_{N+1}(\xi,\eta) \in S^{\mu-(N+1)}(\mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H}),$$

(See the formula (2.3).) Then the above computation for $\mu - j$ rather than μ, using that

$$\chi(\delta\xi,\delta\eta)a_{(\mu-j)}(\delta\xi,\delta\eta) = \delta^{\mu-j}\tilde{\kappa}_{\delta}\chi(\xi,\eta)a_{(\mu-j)}(\xi,\eta)\kappa_{\delta}^{-1}$$

for $\delta \geq 1$ and $|\eta|$ large enough, gives us for

$$p_{\mu-j}(\eta) = \mathrm{Op}_x(\chi a_{(\mu-j)})(\eta)$$

the relation

$$p_{\mu-j}(\delta\eta) = \delta^{\mu-j}\tilde{\chi}_\delta p_{\mu-j}(\eta)\chi_\delta^{-1}$$

for $\delta \geq 1$ and $|\eta|$ large. (See also Example 2.1.) Thus for every $N \in \mathbb{N}$ we obtain

$$p(\eta) = \sum_{j=0}^{N} p_{\mu-j}(\eta) + p_{N+1}(\eta)$$

for

$$p_{N+1}(\eta) = \mathrm{Op}_x(r_{N+1})(\eta) \in S^{\mu-(N+1)}(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}}.$$

(See Theorem 3.1.) Thus $p(\eta)$ is a classical symbol. □

Proposition 3.3. *The correspondence $a(\xi,\eta) \mapsto p(\eta)$ induces a continuous operator*

$$S^\mu_{(\mathrm{cl})}(\mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}} \to S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_\eta; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}},$$

for all $s \in \mathbb{R}$.

Proof. The continuity for general symbols is an immediate consequence of the proof of Theorem 3.1. The assertion in the classical case is straightforward for the terms that are homogeneous for large absolute values of covariables. The arguments for the remainder terms are as in the first part of the proof. □

Remark 3.4. (i) More generally, let $a(x,y,\xi,\eta) \in \mathcal{S}(\mathbb{R}^p_x, S^\mu(\mathbb{R}^q_y \times \mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}})$ and let $p(y,\eta) = \mathrm{Op}_x(a)(y,\eta)$. Then we have

$$p(y,\eta) \in S^\mu(\mathbb{R}^q_y \times \mathbb{R}^q_\eta; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}}$$

for every $s \in \mathbb{R}$, and the map $a \mapsto p$ induces a continuous operator

$$\mathcal{S}(\mathbb{R}^p_x, S^\mu(\mathbb{R}^q_y \times \mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}}) \to S^\mu(\mathbb{R}^q_y \times \mathbb{R}^q_\eta; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}}.$$

(ii) If $a(y,\xi,\eta) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^q_y \times \mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}}$ and $p(y,\eta) = \mathrm{Op}_x(a)(y,\eta)$, then we have

$$p(y,\eta) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}},$$

and $a \mapsto p$ defines a continuous operator between the respective symbol spaces.

Let H_1 be a Hilbert space with group action κ_1. Then $H_2 = \mathcal{W}^s(\mathbb{R}^{q_1}, H_1)$ for $q_1 \in \mathbb{N}$, admits the group action κ_2 in accordance with

$$(\kappa_{2,\delta}u_2)(y^1) = \delta^{q_1/2}(\kappa_{1,\delta}u_2)(\delta y^1), \quad y^1 \in \mathbb{R}^{q_1},\ \delta \in \mathbb{R}_+.$$

(See Theorem 1.2.) If we let

$$H_k = \mathcal{W}^s(\mathbb{R}^{q_{k-1}}, H_{k-1}) \tag{3.9}$$

for $k \geq 2, q_{k-1} \in \mathbb{N}$, then we have the group action κ_k on H_k given in (3.9) defined by

$$(\kappa_{k,\delta} u_k)(y^{k-1}) = \delta^{q_{k-1}/2}(\kappa_{k-1,\delta} u_k)(\delta y^{k-1}), \quad y^{k-1} \in \mathbb{R}^{q_{k-1}}.$$

As a corollary of Theorem 3.1, we obtain the following iteration.

Corollary 3.5. *Let* $a(\xi, \eta^1, \eta^2, \ldots, \eta^k) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^p_\xi \times \mathbb{R}^{q_1} \times \ldots \times \mathbb{R}^{q_k}; H_0, \widetilde{H}_0)_{\chi_0, \tilde{\chi}_0}$, *where* $H_0, \widetilde{H}_0$ *and* $\chi_0, \tilde{\chi}_0$ *are as in Theorem* 3.1*. For* $s \in \mathbb{R}$*, let*

$$H_1 = \mathcal{W}^s(\mathbb{R}^p_x, H_0)_{\chi_0},\ \widetilde{H}_1 = \mathcal{W}^{s-\mu}(\mathbb{R}^p_x, \widetilde{H}_0)_{\tilde{\chi}_0},$$

$s \in \mathbb{R}$*, with group actions*

$$\chi_{1,\delta} f_1(x) = \delta^{p/2}(\chi_{0,\delta} f_1)(\delta x),\ \tilde{\chi}_{1,\delta} \tilde{f}_1(x) = \delta^{p/2}(\tilde{\chi}_{0,\delta} \tilde{f}_1)(\delta x), \quad \delta \in \mathbb{R}_+.$$

Then by Theorem 3.1*, we have*

$$a_1(\eta^1, \eta^2, \ldots, \eta^k) = \mathrm{Op}_x(a)(\eta^1, \eta^2, \ldots, \eta^k) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \times \ldots \times \mathbb{R}^{q_k}; H_1, \widetilde{H}_1)_{\chi_1, \tilde{\chi}_1}.$$

For $2 \leq l \leq k-1$ *and*

$$H_l = \mathcal{W}^s(\mathbb{R}^{q_{l-1}}, H_{l-1})_{\chi_{l-1}},\ \widetilde{H}_l = \mathcal{W}^{s-\mu}(\mathbb{R}^{q_{l-1}}, \widetilde{H}_{l-1})_{\tilde{\chi}_{l-1}}$$

endowed with group actions $\chi_l, \tilde{\chi}_l$ *defined by*

$$\chi_{l,\delta} f_l(y^{l-1}) = \delta^{q_{l-1}/2}(\chi_{l-1,\delta} f_l)(\delta y^{l-1}),\ \tilde{\chi}_{l,\delta} \tilde{f}_l(y^{l-1}) = \delta^{q_{l-1}/2}(\tilde{\chi}_{l-1,\delta} \tilde{f}_l)(\delta y^{l-1}),$$

we have

$$a_l(\eta^l, \ldots, \eta^k) = \mathrm{Op}_{y^{l-1}}(a_{l-1})(\eta^l, \ldots, \eta^k) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q_l} \times \ldots \times \mathbb{R}^{q_k}; H_l, \widetilde{H}_l)_{\chi_l, \tilde{\chi}_l}.$$

The correspondence $a(\xi, \eta^1, \ldots, \eta^k) \mapsto a_k(\eta^k)$ induces a continuous operator

$$S^\mu_{(\mathrm{cl})}(\mathbb{R}^p_\xi \times \mathbb{R}^{q_1} \times \ldots \times \mathbb{R}^{q_k}; H, \widetilde{H})_{\kappa, \tilde{\kappa}} \to S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q_k}; H_k, \widetilde{H}_k)_{\chi_k, \tilde{\chi}_k}$$

for every $k \in \mathbb{N}, s \in \mathbb{R}$.
The following proposition is related with the shape of decoupled symbols (2.16).

Proposition 3.6. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^p_x)$ *and* $\mathcal{M}_\varphi$ *the operator of multiplication by* φ*. Then for every* $s \in \mathbb{R}$*, we have* $\mathcal{M}_\varphi \in S^0(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^s(\mathbb{R}^p, H)_\kappa)_{\chi,\chi}$*, and* $\varphi \to \mathcal{M}_\varphi$ *gives rise to a continuous injective operator*

$$\mathcal{S}(\mathbb{R}^p) \to S^0(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^s(\mathbb{R}^p, H)_\kappa)_{\chi,\chi}. \tag{3.10}$$

Proof. We have to verify the symbolic estimates

$$\|\chi_{\langle\eta\rangle}^{-1}\mathcal{M}_\varphi\chi_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^p,H))} \le c \tag{3.11}$$

for a $c = c_\varphi > 0$ and $c_\varphi \to 0$ as $\varphi \to 0$ in $\mathcal{S}(\mathbb{R}^p_x)$. We have

$$\chi_{\langle\eta\rangle}^{-1}\mathcal{M}_\varphi\chi_{\langle\eta\rangle} = \mathcal{M}_{\varphi_\eta}$$

for $\varphi_\eta(x) = \varphi(\langle\eta\rangle^{-1}x)$. We now employ the fact that for the scalar symbol $\langle\xi\rangle^s$, the operator $\mathrm{Op}(\langle\xi\rangle^s)$ for $\mathrm{Op} = \mathrm{Op}_x$ induces an isomorphism

$$\mathrm{Op}(\langle\xi\rangle^s) : \mathcal{W}^s(\mathbb{R}^p, H) \to \mathcal{W}^0(\mathbb{R}^p, H)$$

with the inverse $\mathrm{Op}(\langle\xi\rangle^{-s})$. We have

$$\begin{aligned}\mathrm{Op}(\mathcal{M}_{\varphi_\eta}) &= \mathrm{Op}(\langle\xi\rangle^{-s})\mathrm{Op}(\langle\xi\rangle^s)\mathrm{Op}(\mathcal{M}_{\varphi_\eta})\mathrm{Op}(\langle\xi\rangle^{-s})\mathrm{Op}(\langle\xi\rangle^s)\\ &= \mathrm{Op}(\langle\xi\rangle^{-s})\mathrm{Op}(f_\eta)\mathrm{Op}(\langle\xi\rangle^s),\end{aligned} \tag{3.12}$$

where $f_\eta(x,\xi) = \langle\xi\rangle^s\#(\mathcal{M}_{\varphi_\eta}\langle\xi\rangle^{-s})$ and $\#$ is the Leibniz product between symbols in (x,ξ). Now, we show that

$$\|\mathrm{Op}(f_\eta)\|_{\mathcal{L}(\mathcal{W}^0(\mathbb{R}^p,H))} \le c$$

for some $c = c_\varphi > 0$ independent of $\eta \in \mathbb{R}^q$ and $c_\varphi \to 0$ as $\varphi \to 0$ in $\mathcal{S}(\mathbb{R}^p_x)$. Writing for the moment $a(\xi) = \langle\xi\rangle^s$, $b(x,\xi) = \varphi_\eta(x)\langle\xi\rangle^{-s}$, we have

$$f_\eta(x,\xi) = (a\#b)(x,\xi)$$

after Kumano-go's formalism [16] as an oscillatory integral, namely,

$$f_\eta(x,\xi) = \langle\xi\rangle^{-s}\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{-iz\zeta}\langle\xi+\zeta\rangle^s\varphi(\langle\eta\rangle^{-1}(x+z))dz đ\zeta.$$

Applying a regularization argument, we obtain

$$\begin{aligned}&\langle\xi\rangle^s f_\eta(x,\xi)\\ = &\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{-iz\zeta}\langle\zeta\rangle^{-2M}(1-\Delta_\zeta)^N\langle\xi+\zeta\rangle^s\langle z\rangle^{-2N}(1-\Delta_z)^M\varphi(\langle\eta\rangle^{-1}(x+z))dz đ\zeta\end{aligned}$$

for any $M, N \in \mathbb{N}$. We choose M, N in such a way that $|s| - 2M < -p, -2N < -p$. Then it follows that

$$\begin{aligned}&|f_\eta(x,\xi)|\\ \le\ &\langle\xi\rangle^{-s}\int_{\mathbb{R}^p}\langle\zeta\rangle^{-2M}(1-\Delta_\zeta)^N\langle\xi+\zeta\rangle^s đ\zeta\int_{\mathbb{R}^p}|\langle z\rangle^{-2N}(1-\Delta_z)^M\varphi(\langle\eta\rangle^{-1}(x+z))|dz.\end{aligned} \tag{3.13}$$

Let us first consider the expression

$$F(\xi) = \langle\xi\rangle^{-s} \int_{\mathbb{R}^p} \langle\zeta\rangle^{-2M} (1-\Delta_\zeta)^N \langle\xi+\zeta\rangle^s đ\zeta.$$

We have

$$(1-\Delta_\zeta)^N \langle\xi+\zeta\rangle^s = \sum_{j=0}^{2N} \langle\xi+\zeta\rangle^{s-j} B_j(\xi,\zeta)$$

for functions $B_j(\xi,\zeta)$ satisfying the estimates

$$\sup_{\xi,\zeta\in\mathbb{R}^p} |B_j(\xi,\zeta)| \le \text{constant}$$

for different positive constants. This implies that

$$|F(\xi)| \le c\langle\xi\rangle^{-s} \sum_{j=0}^{2N} \int_{\mathbb{R}^p} \langle\zeta\rangle^{-2M} \langle\xi+\zeta\rangle^{s-j} đ\zeta.$$

Using Peetre's inequality to the effect that $\langle\xi+\zeta\rangle^t \le C\langle\xi\rangle^t\langle\zeta\rangle^{|t|}$, we get

$$\langle\xi\rangle^{-s} \int \langle\zeta\rangle^{-2M} \langle\xi+\zeta\rangle^{s-j} đ\zeta \le C\langle\zeta\rangle^{-2M+|s-j|} đ\zeta \le C'$$

for some $C' > 0$. Thus, $|F(\xi)| \le c$ for some $c > 0$. Moreover, $\langle z\rangle^{-2N}(1-\Delta_z)^M \varphi(\langle\eta\rangle^{-1}(x+z))$ is a finite linear combination of terms of the form

$$\langle z\rangle^{-2N} D_z^\alpha \varphi(\langle\eta\rangle^{-1}(x+z)) = \langle\eta\rangle^{-|\alpha|} \langle z\rangle^{-2N} (D_z^\alpha\varphi)(\langle\eta\rangle^{-1}(x+z)),\ |\alpha| \le 2M.$$

Thus, the second factor on the right hand side of (3.13) is a finite linear combination of integrals

$$\langle\eta\rangle^{-|\alpha|} \int \langle z\rangle^{-2N} |(D_z^\alpha\varphi)(\langle\eta\rangle^{-1}(x+z))| dz,\ |\alpha| \le 2M. \tag{3.14}$$

For $\psi_\alpha(\langle\eta\rangle^{-1}(x+z)) = (D_z^\alpha\varphi)(\langle\eta\rangle^{-1}(x+z))$ we obtain by substituting $v = c^{-1}(x+z)$ for $c = \langle\eta\rangle$,

$$\begin{aligned}\int_{\mathbb{R}^p} \langle z\rangle^{-2N} |\psi_\alpha(\langle\eta\rangle^{-1}(x+z))| dz &= \int_{\mathbb{R}^p} \langle cv-x\rangle^{-2N} |\psi_\alpha(v)| c^p dv \\ &\le \sup_{v\in\mathbb{R}^p} |\psi_\alpha(v)| \int_{\mathbb{R}^p} \langle cv-x\rangle^{-2N} c^p dv.\end{aligned}$$

Let $w = cv$. Then we obtain

$$\int_{\mathbb{R}^p} \langle cv-x\rangle^{-2N} c^p dv = \int_{\mathbb{R}^p} \langle w-x\rangle^{-2N} dw = \int_{\mathbb{R}^p} \langle w\rangle^{-2N} dw = \text{constant}.$$

It follows that

$$|f_\eta(x,\xi)| \le c \sum_{|\alpha|\le 2M} \sup_{x\in\mathbb{R}^p} |D_x^\alpha \varphi(x)|.$$

In a similar manner, we see that for any fixed $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^p$ there is a semi-norm $\nu(\cdot)$ on the space $\mathcal{S}(\mathbb{R}^p)$ such that

$$\sup\{|D_x^\alpha D_\xi^\beta f_\eta(x,\xi)| : (x,\xi)\in\mathbb{R}^{2p}, \alpha\le\boldsymbol{\alpha}, \beta\le\boldsymbol{\beta}\} \le c\nu(\varphi)$$

for a positive constant c, which is independent of $\eta\in\mathbb{R}^q$. Let us now recall a version of Calderón-Vaillancourt's theorem, proved by Seiler [32]. (See also Hwang's paper [13] for the scalar case.) Let H and $\widetilde{H}$ be Hilbert spaces with group actions κ and $\tilde\kappa$, respectively, and let $f(x,\xi)\in C^\infty(\mathbb{R}^{2p},\mathcal{L}(H,\widetilde{H}))$ be a function such that

$$\pi(f) = \sup\{\|\tilde\kappa_{\langle\xi\rangle}^{-1}\{D_x^\alpha D_\xi^\beta f_\eta(x,\xi)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} : (x,\xi)\in\mathbb{R}^{2p}, \alpha\le\boldsymbol{\alpha}, \beta\le\boldsymbol{\beta}\}$$

is finite for $\boldsymbol{\alpha} = (g_{\tilde\kappa}+1,\dots,g_{\tilde\kappa}+1)$, $\boldsymbol{\beta} = (1,\dots,1)$, where $g_{\tilde\kappa}$ is the constant in (1.1) corresponding to $\tilde\kappa$. Then

$$\mathrm{Op}(f) : \mathcal{W}^0(\mathbb{R}^p, H) \to \mathcal{W}^0(\mathbb{R}^p,\widetilde{H})$$

is continuous, and we have

$$\|\mathrm{Op}(f)\|_{\mathcal{L}(\mathcal{W}^0(\mathbb{R}^p,H),\mathcal{W}^0(\mathbb{R}^p,\widetilde{H}))} \le c\pi(f)$$

for a positive constant $c>0$ independent of f. We apply this theorem to the case $H=\widetilde{H}$, $\kappa=\tilde\kappa$, and to $f(x,\xi) = f_\eta(x,\xi)\cdot \mathrm{id}_H$. Then for $\boldsymbol{\alpha}=\boldsymbol{\beta}=(1,\dots,1)$, we have

$$\pi(f_\eta) = \sup\{|D_x^\alpha D_\xi^\beta f_\eta(x,\xi)| : (x,\xi)\in\mathbb{R}^{2p}, \alpha\le\boldsymbol{\alpha}, \beta\le\boldsymbol{\beta}\} \le c\nu(\varphi).$$

It follows that

$$\|\mathrm{Op}(f_\eta)\|_{\mathcal{L}(\mathcal{W}^0(\mathbb{R}^p,H))} \le c\nu(\varphi).$$

Applying (3.12), we can return to (3.11) for an arbitrary $s\in\mathbb{R}$ and obtain

$$\sup_{\eta\in\mathbb{R}^q} \|\chi_{\langle\eta\rangle}^{-1}\mathcal{M}_\varphi \chi_{\langle\eta\rangle}\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^p,H))} \le c\nu(\varphi).$$

This is the only semi-norm in $S^0(\mathbb{R}^q;\mathcal{W}^s(\mathbb{R}^p,H)_\kappa,\mathcal{W}^s(\mathbb{R}^p,H)_\kappa)_{\chi,\chi}$ that we have to control. Summing up, we have proved that $\mathcal{M}_\varphi$ is a symbol as claimed and the continuity of (3.10). □

The following theorem extends Theorem 3.1 to symbols with variable coefficients.

Theorem 3.7. *Let $H, \widetilde{H}$ and $\kappa, \tilde{\kappa}$ be as in Theorem* 3.1. *Then for*

$$a(x,y,\xi,\eta) \in \mathcal{S}(\mathbb{R}^p_x, S^\mu(\mathbb{R}^q_y \times \mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}})$$

and

$$p(y,\eta) = \mathrm{Op}_x(a)(y,\eta),$$

we have

$$p(y,\eta) \in S^\mu(\mathbb{R}^q_y \times \mathbb{R}^q_\eta; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}} \tag{3.15}$$

for every $s \in \mathbb{R}$.

Proof. Let us first observe that the extension of Theorem 3.1 to symbols

$$a(y,\xi,\eta) \in S^\mu(\mathbb{R}^q_y \times \mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}}$$

is straightforward because the additional variable y as an action from the left does not influence the proof, and we obtain (3.15) in this case. Therefore, without loss of generality, we omit y and only consider the case

$$a(x,\xi,\eta) \in \mathcal{S}(\mathbb{R}^p_x)\hat{\otimes}_\pi E$$

for $E = S^\mu(\mathbb{R}^{p+q}_{\xi,\eta}; H, \widetilde{H})_{\kappa,\tilde{\kappa}}$. On the right hand side, we use the fact that

$$\mathcal{S}(\mathbb{R}^p, E) = \mathcal{S}(\mathbb{R}^p)\hat{\otimes}_\pi E$$

for the Fréchet space E. As in the proof of Proposition 2.3 we employ a tensor product argument. We write $a(x,\xi,\eta)$ as a convergent sum

$$a(x,\xi,\eta) = \sum_{j=0}^{\infty} \lambda_j \mathcal{M}_{\varphi_j} a_j(\xi,\eta)$$

for sequences $\varphi_j \in \mathcal{S}(\mathbb{R}^p)$, $a_j \in E$, tending to zero in the respective spaces as $j \to \infty$, and $\lambda_j \in \mathbb{C}$, $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. From Proposition 3.3 we conclude that

$$\mathrm{Op}_x(a_j)(\eta) \to 0$$

in $S^\mu(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}}$ and from Proposition 3.6 that

$$\mathrm{Op}_x(\mathcal{M}_{\varphi_j}) \to 0$$

in $S^0(\mathbb{R}^q; \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}}, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\tilde{\chi},\tilde{\chi}}$ as $j \to \infty$. The operator $\mathrm{Op}_x(\mathcal{M}_{\varphi_j})$ representing a symbol that is independent of η can be still identified with the operator of multiplication $\mathcal{M}_{\varphi_j}$. Therefore, it is justified to interpret $\mathrm{Op}_x(a)(\eta)$ as a convergent sum

$$\mathrm{Op}_x(a)(\eta) = \sum_{j=0}^{\infty} \lambda_j \mathrm{Op}_x(\mathcal{M}_{\varphi_j}) \mathrm{Op}_x(a_j)(\eta)$$

in $\mathcal{S}(\mathbb{R}^p_x) \hat{\otimes}_\pi S^\mu(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}}$ continuously embedded in $S^\mu(\mathbb{R}^q; \mathcal{W}^s(\mathbb{R}^p, H)_\kappa, \mathcal{W}^{s-\mu}(\mathbb{R}^p, \widetilde{H})_{\tilde{\kappa}})_{\chi,\tilde{\chi}}$. Thus $\mathrm{Op}_x(a)(\eta)$ is an element in the latter symbol space. □

Remark 3.8. As in Corollary 3.5 we can iterate Theorem 3.7 for symbols

$$a(x, y^1, \ldots, y^k, \xi, \eta^1, \eta^2, \ldots, \eta^k) \in \mathcal{S}(\mathbb{R}^p_x \times \mathbb{R}^{q_1+\ldots+q_{k-1}}, S^\mu(\mathbb{R}^{q_k}; H_0, \widetilde{H}_0)_{\chi_0,\tilde{\chi}_0}) \quad (3.16)$$

for $H_0, \widetilde{H}_0$ and $\chi_0, \tilde{\chi}_0$ as in Theorem 3.1. Then we have the same iterative process to successively build up symbols

$$\begin{aligned} a_l(y^l, \ldots, y^k, \eta^l, \ldots, \eta^k) &= \mathrm{Op}_{y^{l-1}}(a_{l-1})(y^l, \ldots, y^k, \eta^l, \ldots, \eta^k) \\ &\in \mathcal{S}(\mathbb{R}^{q_l+\ldots+q_{k-1}}, S^\mu(\mathbb{R}^{q_k} \times \mathbb{R}^{q_l+\ldots+q_k}; H_l, \widetilde{H}_l)_{\chi_l,\tilde{\chi}_l}) \end{aligned}$$

for every $1 \leq l \leq k$, starting with $a_0 = a$ defined by (3.16), then $a_1 = \mathrm{Op}_x(a_0)$, and so on.

References

[1] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, *Acta Math.* **126** (1971), 11–51.

[2] D.-C. Chang, N. Habal and B.-W. Schulze, Quantisation on a manifold with singular edge, *J. Pseudo-Differ. Oper. Appl.*, **4** (3) (2013), pp. 317–343.

[3] N. Dines, *Ellipticity of a class of corner operators*, in Pseudo-differential operators: PDE and time-frequency analysis, Fields Inst. Commun. 52, Amer. Math. Soc., Providence, RI, 2007, pp. 131-169.

[4] Ch. Dorschfeldt, *Algebras of pseudo-differential operators near edge and corner singularities*, Math. Res. 102, Wiley-VCH, Berlin, Weinheim, 1998.

[5] M. Dreher and I. Witt, Edge Sobolev spaces and weakly hyperbolic operators, *Ann. Mat. Pura Appl.* **180** (2002), 451-482.

[6] Ju. V. Egorov and B.-W. Schulze, *Pseudo-Differential Operators, Singularities, Applications*, Oper. Theory: Adv. Appl. **93**, Birkhäuser Verlag, Basel, 1997.

[7] G. I. Eskin, *Boundary Value Problems for Elliptic Pseudo-Differential Equations*, Transl. Nauka, Moskva, 1973, Math. Monographs **24** Amer. Math. Soc., 1980.

[8] H.-J. Flad and G. Harutyunyan, *Ellipticity of quantum mechanical Hamiltonians in the edge algebra*, Proc. AIMS Conference on Dynamical Systems, Differential Equations and Applications, Dresden, 2010.

[9] J. B. Gil, B.-W. Schulze, and J. Seiler, Cone pseudodifferential operators in the edge symbolic calculus, *Osaka J. Math.* **37** (2000), 219–258.

[10] N. Habal and B.-W. Schulze, *Mellin quantisation in corner operators*, Operator Theory: Advances and Applications, Vol. 228, (Y.I. Karlovich et. al. eds.), "Operator Theory, Pseudo-Differential Equations, and Mathematical Physics", The Vladimir Rabinovich Anniversary Volume, Birkhäuser, Basel, 2013, pp. 151–172.

[11] G. Harutyunyan and B.-W. Schulze, *Elliptic Mixed, Transmission and Singular Crack Problems*, European Mathematical Soc., Zürich, 2008.

[12] T. Hirschmann, Functional analysis in cone and edge Sobolev spaces, *Ann. Global Anal. Geom.* **8** (1990), 167–192.

[13] I. L. Hwang, The L^2-boundedness of pseudodifferential operators, *Trans. Amer. Math. Soc.* **302** (1987), 55-76.

[14] V. A. Kondratyev, Boundary value problems for elliptic equations in domains with conical points, *Trudy Mosk. Mat. Obshch.* **16** (1967), 209–292.

[15] V. A. Kondratyev and O.A. Oleynik, Boundary problems for partial differential equations on non-smooth domains, *Uspekhi Mat. Nauk* **38** (1983), 3–76.

[16] H. Kumano-go, *Pseudo-differential operators*, The MIT Press, Cambridge, Massachusetts and London, England, 1981.

[17] X. Liu and B.-W. Schulze, Ellipticity on manifolds with edges and boundary, *Monatsh. Math.* **146** (2005), 295–331.

[18] G. Luke, Pseudo-differential operators on Hilbert bundles, *J. Differential Equations* **12** (1972), 566-589.

573–592.

[19] V. S. Rabinovich, Pseudo-differential operators in non-bounded domains with conical structure at infinity, *Mat. Sb.* **80** (1969), 77–97.

[20] S. Rempel and B.-W. Schulze, Complete Mellin and Green symbolic calculus in spaces with conormal asymptotics, *Ann. Global Anal. Geom.* **4** (1986), 137–224.

[21] S. Rempel and B.-W. Schulze, *Asymptotics for elliptic mixed boundary problems* (*pseudo-differential and Mellin operators in spaces with conormal singularity*), Math. Res. **50**, Akademie-Verlag, Berlin, 1989.

[22] W. Rungrottheera, Parameter-dependent corner operators, *Asian-Eur. J. Math.* **6** (2013), 1–29.

[23] W. Rungrottheera and B.-W. Schulze, Weighted spaces on corner manifolds, *Complex Var. Elliptic Equ.*, doi/full/10.1080/17476933.2013.876416.

[24] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, in: Symp Partial Differential Equations. Holzhau 1988, Teubner-Texte zur Mathematik Vol. 112, Teubner, Leipzig, 1989, pp. 259–287.

[25] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

[26] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, J. Wiley, Chichester, 1998.

[27] B.-W. Schulze, Operators with symbol hierarchies and iterated asymptotics, *Publ. Res. Inst. Math. Sci.*, Kyoto **38** (2002), 735–802.

[28] B.-W. Schulze, *The iterative structure of the corner calculus*, Oper. Theory: Adv. Appl. **213**, Pseudo-Differential Operators: Analysis, Application and Computations (L. Rodino, M. W. Wong and H. Zhu, eds.), Birkhäuser Verlag, Basel, 2011, pp. 79–103.

[29] B.-W. Schulze and J. Seiler, The edge algebra structure of boundary value problems, *Ann. Glob. Anal. Geom.* **22** (2002), 197–265.

[30] B.-W. Schulze and M. W. Wong, Mellin operators with asymptotics on manifolds with corners, Oper. Theory: Adv. Appl. **213**, Pseudo-Differential Operators: Analysis, Applications and Computations (L. Rodino, M. W. Wong and H. Zhu, eds.), Birkhäuser Verlag, Basel, 2011, pp. 31–78.

[31] B.-W. Schulze and M. W. Wong, Mellin and Green operators of the corner calculus, *J. Pseudo-Differ. Oper. Appl.* **2** (2011), 467–507.

[32] J. Seiler, Continuity of edge and corner pseudo-differential operators, *Math Nachr.* **205** (1999), 163–182.

[33] M. I. Vishik and V. V. Grushin, On a class of degenerate elliptic equations of higher orders, *Mat. Sb.* **79** (1969), 336.